Chapter 10

Rotation of a Rigid Object About a Fixed Axis

CHAPTER OUTLINE

▲ *The Malaysian pastime of* gasing *involves the spinning of tops that can have masses up to 20 kg. Professional spinners can spin their tops so that they might rotate for hours before stopping. We will study the rotational motion of objects such as these tops in this chapter. (Courtesy Tourism Malaysia)*

When an extended object such as a wheel rotates about its axis, the motion cannot be analyzed by treating the object as a particle because at any given time different parts of the object have different linear velocities and linear accelerations. We can, however, analyze the motion by considering an extended object to be composed of a collection of particles, each of which has its own linear velocity and linear acceleration.

In dealing with a rotating object, analysis is greatly simplified by assuming that the object is rigid. A **rigid object** is one that is nondeformable—that is, the relative locations of all particles of which the object is composed remain constant. All real objects are deformable to some extent; however, our rigid-object model is useful in many situations in which deformation is negligible.

Rigid object

10.1 Angular Position, Velocity, and Acceleration

Figure 10.1 illustrates an overhead view of a rotating compact disc. The disc is rotating about a fixed axis through O. The axis is perpendicular to the plane of the figure. Let us investigate the motion of only one of the millions of "particles" making up the disc. A particle at P is at a fixed distance r from the origin and rotates about it in a circle of radius r. (In fact, *every* particle on the disc undergoes circular motion about O.) It is convenient to represent the position of P with its polar coordinates (r, θ), where r is the distance from the origin to P and θ is measured *counterclockwise* from some reference line as shown in Figure 10.1a. In this representation, the only coordinate for the particle that changes in time is the angle θ; r remains constant. As the particle moves along the circle from the reference line ($\theta = 0$), it moves through an arc of length s, as in Figure 10.1b. The arc length s is related to the angle θ through the relationship

$$s = r\theta \tag{10.1a}$$

$$\theta = \frac{s}{r} \tag{10.1b}$$

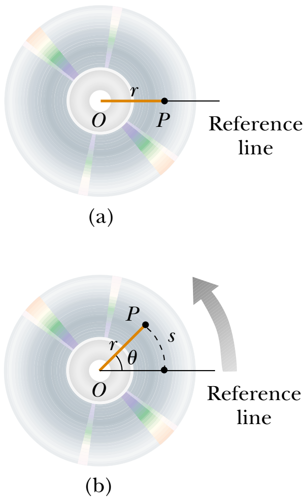

Figure 10.1 A compact disc rotating about a fixed axis through O perpendicular to the plane of the figure. (a) In order to define angular position for the disc, a fixed reference line is chosen. A particle at P is located at a distance r from the rotation axis at O. (b) As the disc rotates, point P moves through an arc length s on a circular path of radius r.

Note the dimensions of θ in Equation 10.1b. Because θ is the ratio of an arc length and the radius of the circle, it is a pure number. However, we commonly give θ the artificial unit **radian** (rad), where

one radian is the angle subtended by an arc length equal to the radius of the arc.

Because the circumference of a circle is $2\pi r$, it follows from Equation 10.1b that 360° corresponds to an angle of $(2\pi r/r)$ rad $= 2\pi$ rad. (Also note that 2π rad corresponds

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10.1 Remember the Radian

In rotational equations, we must use angles expressed in *radians.* Don't fall into the trap of using angles measured in degrees in rotational equations.

to one complete revolution.) Hence, 1 rad $= 360°/2\pi \approx 57.3°$. To convert an angle in degrees to an angle in radians, we use the fact that π rad $= 180°$, or

$$\theta\,(\text{rad}) = \frac{\pi}{180°}\,\theta\,(\text{deg})$$

For example, 60° equals $\pi/3$ rad and 45° equals $\pi/4$ rad.

Because the disc in Figure 10.1 is a rigid object, as the particle moves along the circle from the reference line, every other particle on the object rotates through the same angle θ. Thus, **we can associate the angle θ with the entire rigid object as well as with an individual particle.** This allows us to define the *angular position* of a rigid object in its rotational motion. We choose a reference line on the object, such as a line connecting O and a chosen particle on the object. The **angular position** of the rigid object is the angle θ between this reference line on the object and the fixed reference line in space, which is often chosen as the x axis. This is similar to the way we identify the position of an object in translational motion—the distance x between the object and the reference position, which is the origin, $x = 0$.

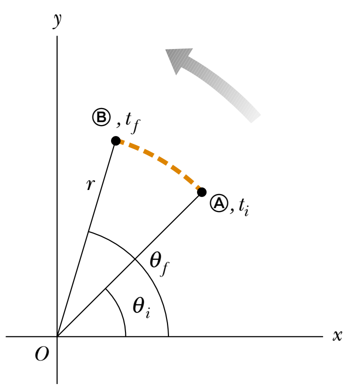

Figure 10.2 A particle on a rotating rigid object moves from Ⓐ to Ⓑ along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radius vector moves through an angular displacement $\Delta\theta = \theta_f - \theta_i$.

As the particle in question on our rigid object travels from position Ⓐ to position Ⓑ in a time interval Δt as in Figure 10.2, the reference line of length r sweeps out an angle $\Delta\theta = \theta_f - \theta_i$. This quantity $\Delta\theta$ is defined as the **angular displacement** of the rigid object:

$$\Delta\theta \equiv \theta_f - \theta_i$$

The rate at which this angular displacement occurs can vary. If the rigid object spins rapidly, this displacement can occur in a short time interval. If it rotates slowly, this displacement occurs in a longer time interval. These different rotation rates can be quantified by introducing *angular speed.* We define the **average angular speed** $\overline{\omega}$ (Greek omega) as the ratio of the angular displacement of a rigid object to the time interval Δt during which the displacement occurs:

Average angular speed

$$\overline{\omega} \equiv \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad (10.2)$$

In analogy to linear speed, the **instantaneous angular speed** ω is defined as the limit of the ratio $\Delta\theta/\Delta t$ as Δt approaches zero:

Instantaneous angular speed

$$\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (10.3)$$

Angular speed has units of radians per second (rad/s), which can be written as second^{-1} (s^{-1}) because radians are not dimensional. We take ω to be positive when θ is increasing (counterclockwise motion in Figure 10.2) and negative when θ is decreasing (clockwise motion in Figure 10.2).

Quick Quiz 10.1 A rigid object is rotating in a counterclockwise sense around a fixed axis. Each of the following pairs of quantities represents an initial angular position and a final angular position of the rigid object. Which of the sets can *only* occur if the rigid object rotates through more than 180°? (a) 3 rad, 6 rad (b) −1 rad, 1 rad (c) 1 rad, 5 rad.

Quick Quiz 10.2 Suppose that the change in angular position for each of the pairs of values in Quick Quiz 10.1 occurs in 1 s. Which choice represents the lowest average angular speed?

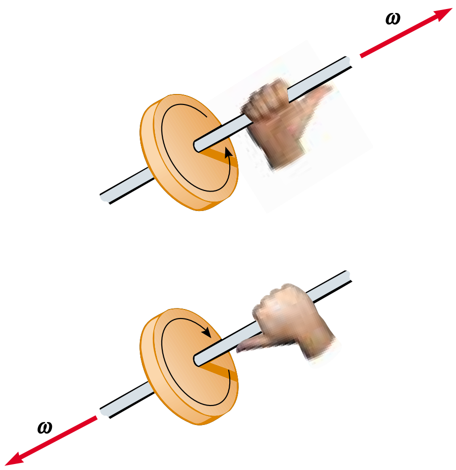

Figure 10.3 The right-hand rule for determining the direction of the angular velocity vector.

If the instantaneous angular speed of an object changes from ω_i to ω_f in the time interval Δt, the object has an angular acceleration. The **average angular acceleration** $\overline{\alpha}$ (Greek alpha) of a rotating rigid object is defined as the ratio of the change in the angular speed to the time interval Δt during which the change in the angular speed occurs:

$$\overline{\alpha} \equiv \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \tag{10.4}$$

Average angular acceleration

In analogy to linear acceleration, the **instantaneous angular acceleration** is defined as the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero:

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \tag{10.5}$$

Instantaneous angular acceleration

Angular acceleration has units of radians per second squared (rad/s^2), or just second^{-2} (s^{-2}). Note that α is positive when a rigid object rotating counterclockwise is speeding up or when a rigid object rotating clockwise is slowing down during some time interval.

When a rigid object is rotating about a *fixed* axis, **every particle on the object rotates through the same angle in a given time interval and has the same angular speed and the same angular acceleration.** That is, the quantities θ, ω, and α characterize the rotational motion of the entire rigid object as well as individual particles in the object. Using these quantities, we can greatly simplify the analysis of rigid-object rotation.

Angular position (θ), angular speed (ω), and angular acceleration (α) are analogous to linear position (x), linear speed (v), and linear acceleration (a). The variables θ, ω, and α differ dimensionally from the variables x, v, and a only by a factor having the unit of length. (See Section 10.3.)

We have not specified any direction for angular speed and angular acceleration. Strictly speaking, ω and α are the magnitudes of the angular velocity and the angular acceleration vectors[1] $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$, respectively, and they should always be positive. Because we are considering rotation about a fixed axis, however, we can use nonvector notation and indicate the directions of the vectors by assigning a positive or negative sign to ω and α, as discussed earlier with regard to Equations 10.3 and 10.5. For rotation about a fixed axis, the only direction that uniquely specifies the rotational motion is the direction along the axis of rotation. Therefore, the directions of $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are along this axis. If an object rotates in the xy plane as in Figure 10.1, the direction of $\boldsymbol{\omega}$ is out of the plane of the diagram when the rotation is counterclockwise and into the plane of the diagram when the rotation is clockwise. To illustrate this convention, it is convenient to use the *right-hand rule* demonstrated in Figure 10.3. When the four fingers of the right

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10.2 Specify Your Axis

In solving rotation problems, you must specify an axis of rotation. This is a new feature not found in our study of translational motion. The choice is arbitrary, but once you make it, you must maintain that choice consistently throughout the problem. In some problems, the physical situation suggests a natural axis, such as the center of an automobile wheel. In other problems, there may not be an obvious choice, and you must exercise judgement.

[1] Although we do not verify it here, the instantaneous angular velocity and instantaneous angular acceleration are vector quantities, but the corresponding average values are not. This is because angular displacements do not add as vector quantities for finite rotations.

hand are wrapped in the direction of rotation, the extended right thumb points in the direction of $\boldsymbol{\omega}$. The direction of $\boldsymbol{\alpha}$ follows from its definition $\boldsymbol{\alpha} \equiv d\boldsymbol{\omega}/dt$. It is in the same direction as $\boldsymbol{\omega}$ if the angular speed is increasing in time, and it is antiparallel to $\boldsymbol{\omega}$ if the angular speed is decreasing in time.

Quick Quiz 10.3 A rigid object is rotating with an angular speed $\omega < 0$. The angular velocity vector $\boldsymbol{\omega}$ and the angular acceleration vector $\boldsymbol{\alpha}$ are antiparallel. The angular speed of the rigid object is (a) clockwise and increasing (b) clockwise and decreasing (c) counterclockwise and increasing (d) counterclockwise and decreasing.

10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

In our study of linear motion, we found that the simplest form of accelerated motion to analyze is motion under constant linear acceleration. Likewise, for rotational motion about a fixed axis, the simplest accelerated motion to analyze is motion under constant angular acceleration. Therefore, we next develop kinematic relationships for this type of motion. If we write Equation 10.5 in the form $d\omega = \alpha\, dt$, and let $t_i = 0$ and $t_f = t$, integrating this expression directly gives

Rotational kinematic equations

$$\omega_f = \omega_i + \alpha t \qquad \text{(for constant } \alpha) \tag{10.6}$$

where ω_i is the angular speed of the rigid object at time $t = 0$. Equation 10.6 allows us to find the angular speed ω_f of the object at any later time t. Substituting Equation 10.6 into Equation 10.3 and integrating once more, we obtain

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \qquad \text{(for constant } \alpha) \tag{10.7}$$

where θ_i is the angular position of the rigid object at time $t = 0$. Equation 10.7 allows us to find the angular position θ_f of the object at any later time t. If we eliminate t from Equations 10.6 and 10.7, we obtain

$$\omega_f^{\,2} = \omega_i^{\,2} + 2\alpha(\theta_f - \theta_i) \qquad \text{(for constant } \alpha) \tag{10.8}$$

This equation allows us to find the angular speed ω_f of the rigid object for any value of its angular position θ_f. If we eliminate α between Equations 10.6 and 10.7, we obtain

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)\,t \qquad \text{(for constant } \alpha) \tag{10.9}$$

Notice that these kinematic expressions for rotational motion under constant angular acceleration are of the same mathematical form as those for linear motion under constant linear acceleration. They can be generated from the equations for linear motion by making the substitutions $x \rightarrow \theta$, $v \rightarrow \omega$, and $a \rightarrow \alpha$. Table 10.1 compares the kinematic equations for rotational and linear motion.

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10.3 Just Like Translation?

Equations 10.6 to 10.9 and Table 10.1 suggest that rotational kinematics is just like translational kinematics. That is almost true, with two key differences: (1) in rotational kinematics, you must specify a rotation axis (per Pitfall Prevention 10.2); (2) in rotational motion, the object keeps returning to its original orientation—thus, you may be asked for the number of revolutions made by a rigid object. This concept has no meaning in translational motion, but is related to $\Delta\theta$, which is analogous to Δx.

Table 10.1

Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

Quick Quiz 10.4 Consider again the pairs of angular positions for the rigid object in Quick Quiz 10.1. If the object starts from rest at the initial angular position, moves counterclockwise with constant angular acceleration, and arrives at the final angular position with the same angular speed in all three cases, for which choice is the angular acceleration the highest?

Example 10.1 Rotating Wheel

A wheel rotates with a constant angular acceleration of 3.50 rad/s^2.

(A) If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, through what angular displacement does the wheel rotate in 2.00 s?

Solution We can use Figure 10.2 to represent the wheel. We arrange Equation 10.7 so that it gives us angular displacement:

$$\Delta\theta = \theta_f - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2$$
$$= (2.00 \text{ rad/s})(2.00 \text{ s}) + \tfrac{1}{2}(3.50 \text{ rad/s}^2)(2.00 \text{ s})^2$$
$$= \boxed{11.0 \text{ rad}} = (11.0 \text{ rad})(57.3°/\text{rad}) = \boxed{630°}$$

(B) Through how many revolutions has the wheel turned during this time interval?

Solution We multiply the angular displacement found in part (A) by a conversion factor to find the number of revolutions:

$$\Delta\theta = 630°\left(\frac{1 \text{ rev}}{360°}\right) = \boxed{1.75 \text{ rev}}$$

(C) What is the angular speed of the wheel at $t = 2.00$ s?

Solution Because the angular acceleration and the angular speed are both positive, our answer must be greater than 2.00 rad/s. Using Equation 10.6, we find

$$\omega_f = \omega_i + \alpha t = 2.00 \text{ rad/s} + (3.50 \text{ rad/s}^2)(2.00 \text{ s})$$
$$= \boxed{9.00 \text{ rad/s}}$$

We could also obtain this result using Equation 10.8 and the results of part (A). Try it!

What If? **Suppose a particle moves along a straight line with a constant acceleration of 3.50 m/s^2. If the velocity of the particle is 2.00 m/s at $t_i = 0$, through what displacement does the particle move in 2.00 s? What is the velocity of the particle at $t = 2.00$ s?**

Answer Notice that these questions are translational analogs to parts (A) and (C) of the original problem. The mathematical solution follows exactly the same form. For the displacement,

$$\Delta x = x_f - x_i = v_i t + \tfrac{1}{2}at^2$$
$$= (2.00 \text{ m/s})(2.00 \text{ s}) + \tfrac{1}{2}(3.50 \text{ m/s}^2)(2.00 \text{ s})^2$$
$$= 11.0 \text{ m}$$

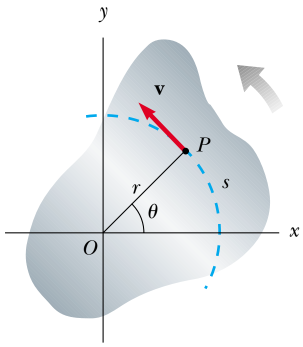
and for the velocity,

$$v_f = v_i + at = 2.00 \text{ m/s} + (3.50 \text{ m/s}^2)(2.00 \text{ s}) = 9.00 \text{ m/s}$$

Note that there is no translational analog to part (B) because translational motion is not repetitive like rotational motion.

10.3 Angular and Linear Quantities

In this section we derive some useful relationships between the angular speed and acceleration of a rotating rigid object and the linear speed and acceleration of a point in the object. To do so, we must keep in mind that when a rigid object rotates about a fixed axis, as in Figure 10.4, **every particle of the object moves in a circle whose center is the axis of rotation.**



Active Figure 10.4 As a rigid object rotates about the fixed axis through O, the point P has a tangential velocity $\mathbf{v}$ that is always tangent to the circular path of radius r.

At the Active Figures link at http://www.pse6.com, you can move point P and observe the tangential velocity as the object rotates.

Because point P in Figure 10.4 moves in a circle, the linear velocity vector $\mathbf{v}$ is always tangent to the circular path and hence is called *tangential velocity*. The magnitude of the tangential velocity of the point P is by definition the tangential speed $v = ds/dt$, where s is the distance traveled by this point measured along the circular path. Recalling that $s = r\theta$ (Eq. 10.1a) and noting that r is constant, we obtain

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Because $d\theta/dt = \omega$ (see Eq. 10.3), we see that

$$v = r\omega \tag{10.10}$$

That is, the tangential speed of a point on a rotating rigid object equals the perpendicular distance of that point from the axis of rotation multiplied by the angular speed. Therefore, although every point on the rigid object has the same *angular* speed, not every point has the same *tangential* speed because r is not the same for all points on the object. Equation 10.10 shows that the tangential speed of a point on the rotating object increases as one moves outward from the center of rotation, as we would intuitively expect. The outer end of a swinging baseball bat moves much faster than the handle.

We can relate the angular acceleration of the rotating rigid object to the tangential acceleration of the point P by taking the time derivative of v:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

Relation between tangential and angular acceleration

$$a_t = r\alpha \tag{10.11}$$

That is, the tangential component of the linear acceleration of a point on a rotating rigid object equals the point's distance from the axis of rotation multiplied by the angular acceleration.

In Section 4.4 we found that a point moving in a circular path undergoes a radial acceleration a_r of magnitude v^2/r directed toward the center of rotation (Fig. 10.5). Because $v = r\omega$ for a point P on a rotating object, we can express the centripetal acceleration at that point in terms of angular speed as

$$a_c = \frac{v^2}{r} = r\omega^2 \tag{10.12}$$

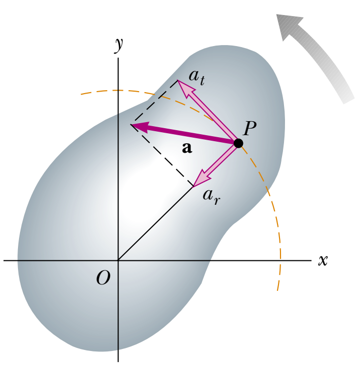


Figure 10.5 As a rigid object rotates about a fixed axis through O, the point P experiences a tangential component of linear acceleration a_t and a radial component of linear acceleration a_r. The total linear acceleration of this point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$.

The total linear acceleration vector at the point is $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r$, where the magnitude of $\mathbf{a}_r$ is the centripetal acceleration a_c. Because $\mathbf{a}$ is a vector having a radial and a tangential component, the magnitude of $\mathbf{a}$ at the point P on the rotating rigid object is

$$a = \sqrt{a_t^2 + a_r^2} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \tag{10.13}$$

Quick Quiz 10.5 Andy and Charlie are riding on a merry-go-round. Andy rides on a horse at the outer rim of the circular platform, twice as far from the center of the circular platform as Charlie, who rides on an inner horse. When the merry-go-round is rotating at a constant angular speed, Andy's angular speed is (a) twice Charlie's (b) the same as Charlie's (c) half of Charlie's (d) impossible to determine.

Quick Quiz 10.6 Consider again the merry-go-round situation in Quick Quiz 10.5. When the merry-go-round is rotating at a constant angular speed, Andy's tangential speed is (a) twice Charlie's (b) the same as Charlie's (c) half of Charlie's (d) impossible to determine.

Example 10.2 CD Player

On a compact disc (Fig. 10.6), audio information is stored in a series of pits and flat areas on the surface of the disc. The information is stored digitally, and the alternations between pits and flat areas on the surface represent binary ones and zeroes to be read by the compact disc player and converted back to sound waves. The pits and flat areas are detected by a system consisting of a laser and lenses. The length of a string of ones and zeroes representing one piece of information is the same everywhere on the disc, whether the information is near the center of the disc or near its outer edge. In order that this length of ones and zeroes always passes by the laser–lens system in the same time period, the tangential speed of the disc surface at the location of the lens must be constant. This requires, according to Equation 10.10, that the angular speed vary as the laser–lens system moves radially along the disc. In a typical compact disc player, the constant speed of the surface at the point of the laser–lens system is 1.3 m/s.

George Semple

Figure 10.6 (Example 10.2) A compact disc.

(A) Find the angular speed of the disc in revolutions per minute when information is being read from the innermost first track ($r = 23$ mm) and the outermost final track ($r = 58$ mm).

Solution Using Equation 10.10, we can find the angular speed that will give us the required tangential speed at the position of the inner track,

$$\omega_i = \frac{v}{r_i} = \frac{1.3 \text{ m/s}}{2.3 \times 10^{-2} \text{ m}} = 57 \text{ rad/s}$$

$$= (57 \text{ rad/s})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right)\left(\frac{60 \text{ s}}{1 \text{ min}}\right)$$

$$= \boxed{5.4 \times 10^2 \text{ rev/min}}$$

For the outer track,

$$\omega_f = \frac{v}{r_f} = \frac{1.3 \text{ m/s}}{5.8 \times 10^{-2} \text{ m}} = 22 \text{ rad/s}$$

$$= \boxed{2.1 \times 10^2 \text{ rev/min}}$$

The player adjusts the angular speed ω of the disc within this range so that information moves past the objective lens at a constant rate.

(B) The maximum playing time of a standard music CD is 74 min and 33 s. How many revolutions does the disc make during that time?

Solution We know that the angular speed is always decreasing, and we assume that it is decreasing steadily, with α constant. If $t = 0$ is the instant that the disc begins, with angular speed of 57 rad/s, then the final value of the time t is $(74 \text{ min})(60 \text{ s/min}) + 33 \text{ s} = 4\,473 \text{ s}$. We are looking for the angular displacement $\Delta\theta$ during this time interval. We use Equation 10.9:

$$\Delta\theta = \theta_f - \theta_i = \tfrac{1}{2}(\omega_i + \omega_f)t$$

$$= \tfrac{1}{2}(57 \text{ rad/s} + 22 \text{ rad/s})(4\,473 \text{ s})$$

$$= 1.8 \times 10^5 \text{ rad}$$

We convert this angular displacement to revolutions:

$$\Delta\theta = 1.8 \times 10^5 \text{ rad}\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = \boxed{2.8 \times 10^4 \text{ rev}}$$

(C) What total length of track moves past the objective lens during this time?

Solution Because we know the (constant) linear velocity and the time interval, this is a straightforward calculation:

$$x_f = v_i t = (1.3 \text{ m/s})(4\,473 \text{ s}) = \boxed{5.8 \times 10^3 \text{ m}}$$

More than 5.8 km of track spins past the objective lens!

(D) What is the angular acceleration of the CD over the 4 473-s time interval? Assume that α is constant.

Solution The most direct approach to solving this problem is to use Equation 10.6 and the results to part (A). We should obtain a negative number for the angular acceleration because the disc spins more and more slowly in the positive direction as time goes on. Our answer should also be relatively small because it takes such a long time—more than an hour—for the change in angular speed to be accomplished:

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{22 \text{ rad/s} - 57 \text{ rad/s}}{4\,473 \text{ s}}$$

$$= \boxed{-7.8 \times 10^{-3} \text{ rad/s}^2}$$

The disc experiences a very gradual decrease in its rotation rate, as expected.

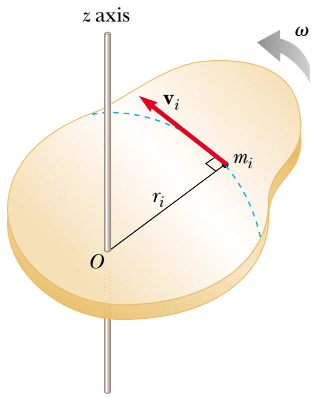


Figure 10.7 A rigid object rotating about the z axis with angular speed ω. The kinetic energy of the particle of mass m_i is $\frac{1}{2}m_iv_i^2$. The total kinetic energy of the object is called its rotational kinetic energy.

10.4 Rotational Kinetic Energy

In Chapter 7, we defined the kinetic energy of an object as the energy associated with its motion through space. An object rotating about a fixed axis remains stationary in space, so there is no kinetic energy associated with translational motion. The individual particles making up the rotating object, however, are moving through space—they follow circular paths. Consequently, there should be kinetic energy associated with rotational motion.

Let us consider an object as a collection of particles and assume that it rotates about a fixed z axis with an angular speed ω. Figure 10.7 shows the rotating object and identifies one particle on the object located at a distance r_i from the rotation axis. Each such particle has kinetic energy determined by its mass and tangential speed. If the mass of the ith particle is m_i and its tangential speed is v_i, its kinetic energy is

$$K_i = \tfrac{1}{2} m_i v_i^2$$

To proceed further, recall that although every particle in the rigid object has the same angular speed ω, the individual tangential speeds depend on the distance r_i from the axis of rotation according to the expression $v_i = r_i\omega$ (see Eq. 10.10). The *total* kinetic energy of the rotating rigid object is the sum of the kinetic energies of the individual particles:

$$K_R = \sum_i K_i = \sum_i \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} \sum_i m_i r_i^2 \omega^2$$

We can write this expression in the form

$$K_R = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \qquad (10.14)$$

where we have factored ω^2 from the sum because it is common to every particle. We simplify this expression by defining the quantity in parentheses as the **moment of inertia *I*:**

Moment of inertia

$$I \equiv \sum_i m_i r_i^2 \qquad (10.15)$$

From the definition of moment of inertia, we see that it has dimensions of ML^2 (kg · m^2 in SI units).[2] With this notation, Equation 10.14 becomes

Rotational kinetic energy

$$K_R = \tfrac{1}{2} I\omega^2 \qquad (10.16)$$

Although we commonly refer to the quantity $\frac{1}{2}I\omega^2$ as **rotational kinetic energy,** it is not a new form of energy. It is ordinary kinetic energy because it is derived from a sum over individual kinetic energies of the particles contained in the rigid object. However, the mathematical form of the kinetic energy given by Equation 10.16 is convenient when we are dealing with rotational motion, provided we know how to calculate I.

It is important that you recognize the analogy between kinetic energy associated with linear motion $\frac{1}{2}mv^2$ and rotational kinetic energy $\frac{1}{2}I\omega^2$. The quantities I and ω in rotational motion are analogous to m and v in linear motion, respectively. (In fact, I takes the place of m and ω takes the place of v every time we compare a linear-motion equation with its rotational counterpart.) The moment of inertia is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion.

PITFALL PREVENTION

10.4 No Single Moment of Inertia

There is one major difference between mass and moment of inertia. Mass is an inherent property of an object. The moment of inertia of an object depends on your choice of rotation axis. Thus, there is no single value of the moment of inertia for an object. There is a minimum value of the moment of inertia, which is that calculated about an axis passing through the center of mass of the object.

[2] Civil engineers use moment of inertia to characterize the elastic properties (rigidity) of such structures as loaded beams. Hence, it is often useful even in a nonrotational context.

Quick Quiz 10.7 A section of hollow pipe and a solid cylinder have the same radius, mass, and length. They both rotate about their long central axes with the same angular speed. Which object has the higher rotational kinetic energy? (a) the hollow pipe (b) the solid cylinder (c) they have the same rotational kinetic energy (d) impossible to determine.

Example 10.3 The Oxygen Molecule

Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length. The mass of each oxygen atom is 2.66×10^{-26} kg, and at room temperature the average separation between the two atoms is $d = 1.21 \times 10^{-10}$ m. (The atoms are modeled as particles.)

(A) Calculate the moment of inertia of the molecule about the z axis.

Solution This is a straightforward application of the definition of I. Because each atom is a distance $d/2$ from the z axis, the moment of inertia about the axis is

$$I = \sum_i m_i r_i^2 = m\left(\frac{d}{2}\right)^2 + m\left(\frac{d}{2}\right)^2 = \frac{md^2}{2}$$

$$= \frac{(2.66 \times 10^{-26}\text{ kg})(1.21 \times 10^{-10}\text{ m})^2}{2}$$

$$= \boxed{1.95 \times 10^{-46}\text{ kg}\cdot\text{m}^2}$$

This is a very small number, consistent with the minuscule masses and distances involved.

(B) If the angular speed of the molecule about the z axis is 4.60×10^{12} rad/s, what is its rotational kinetic energy?

Solution We apply the result we just calculated for the moment of inertia in the equation for K_R:

$$K_R = \tfrac{1}{2} I\omega^2$$

$$= \tfrac{1}{2}(1.95 \times 10^{-46}\text{ kg}\cdot\text{m}^2)(4.60 \times 10^{12}\text{ rad/s})^2$$

$$= \boxed{2.06 \times 10^{-21}\text{ J}}$$

Example 10.4 Four Rotating Objects

Four tiny spheres are fastened to the ends of two rods of negligible mass lying in the xy plane (Fig. 10.8). We shall assume that the radii of the spheres are small compared with the dimensions of the rods.

(A) If the system rotates about the y axis (Fig. 10.8a) with an angular speed ω, find the moment of inertia and the rotational kinetic energy about this axis.

Solution First, note that the two spheres of mass m, which lie on the y axis, do not contribute to I_y (that is, $r_i = 0$ for these spheres about this axis). Applying Equation 10.15, we obtain

$$I_y = \sum_i m_i r_i^2 = Ma^2 + Ma^2 = \boxed{2Ma^2}$$

Therefore, the rotational kinetic energy about the y axis is

$$K_R = \tfrac{1}{2} I_y\omega^2 = \tfrac{1}{2}(2Ma^2)\omega^2 = \boxed{Ma^2\omega^2}$$

The fact that the two spheres of mass m do not enter into this result makes sense because they have no motion about the axis of rotation; hence, they have no rotational kinetic energy. By similar logic, we expect the moment of inertia about the x axis to be $I_x = 2mb^2$ with a rotational kinetic energy about that axis of $K_R = mb^2\omega^2$.

(B) Suppose the system rotates in the xy plane about an axis (the z axis) through O (Fig. 10.8b). Calculate the moment of inertia and rotational kinetic energy about this axis.

Solution Because r_i in Equation 10.15 is the distance between a sphere and the axis of rotation, we obtain

$$I_z = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = \boxed{2Ma^2 + 2mb^2}$$

$$K_R = \tfrac{1}{2} I_z\omega^2 = \tfrac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = \boxed{(Ma^2 + mb^2)\omega^2}$$

Comparing the results for parts (A) and (B), we conclude that the moment of inertia and therefore the rotational kinetic energy associated with a given angular speed depend on the axis of rotation. In part (B), we expect the result to include all four spheres and distances because all four spheres are rotating in the xy plane. Furthermore, the fact that the rotational kinetic energy in part (A) is smaller than that in part (B) indicates, based on the work–kinetic energy theorem, that it would require less work to set the system into rotation about the y axis than about the z axis.

What If? **What if the mass M is much larger than m? How do the answers to parts (A) and (B) compare?**

Answer If $M \gg m$, then m can be neglected and the moment of inertia and rotational kinetic energy in part (B) become

$$I_z = 2Ma^2 \qquad \text{and} \qquad K_R = Ma^2\omega^2$$

which are the same as the answers in part (A). If the masses m of the two red spheres in Figure 10.8 are negligible, then these spheres can be removed from the figure and rotations about the y and z axes are equivalent.

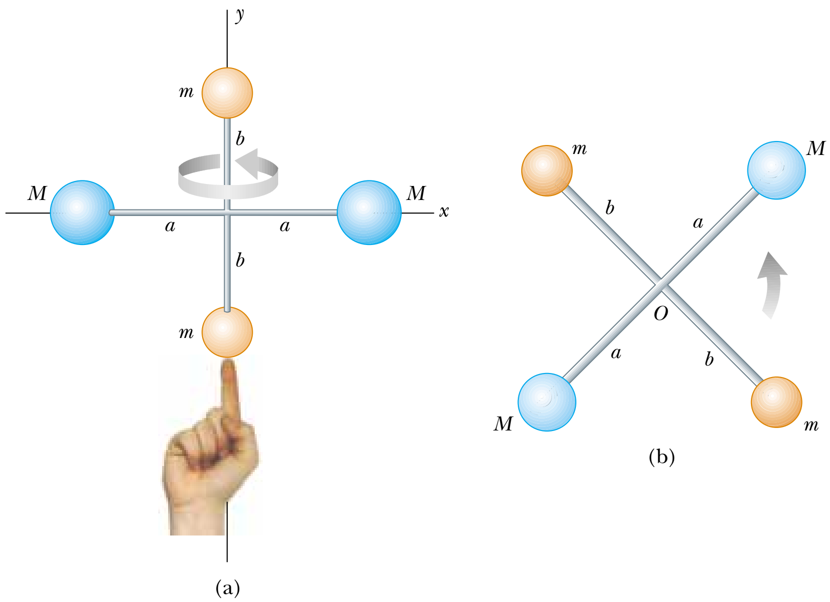


Figure 10.8 (Example 10.4) Four spheres form an unusual baton. (a) The baton is rotated about the y axis. (b) The baton is rotated about the z axis.

10.5 Calculation of Moments of Inertia

We can evaluate the moment of inertia of an extended rigid object by imagining the object to be divided into many small volume elements, each of which has mass Δm_i. We use the definition $I = \sum_i r_i^2 \Delta m_i$ and take the limit of this sum as $\Delta m_i \rightarrow 0$. In this limit, the sum becomes an integral over the volume of the object:

Moment of inertia of a rigid object

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2\, dm \qquad (10.17)$$

It is usually easier to calculate moments of inertia in terms of the volume of the elements rather than their mass, and we can easily make that change by using Equation 1.1, $\rho = m/V$, where ρ is the density of the object and V is its volume. From this equation, the mass of a small element is $dm = \rho\, dV$. Substituting this result into Equation 10.17 gives

$$I = \int \rho r^2\, dV$$

If the object is homogeneous, then ρ is constant and the integral can be evaluated for a known geometry. If ρ is not constant, then its variation with position must be known to complete the integration.

The density given by $\rho = m/V$ sometimes is referred to as *volumetric mass density* because it represents mass per unit volume. Often we use other ways of expressing density. For instance, when dealing with a sheet of uniform thickness t, we can define a *surface mass density* $\sigma = \rho t$, which represents *mass per unit area*. Finally, when mass is distributed along a rod of uniform cross-sectional area A, we sometimes use *linear mass density* $\lambda = M/L = \rho A$, which is the *mass per unit length*.

Example 10.5 Uniform Thin Hoop

Find the moment of inertia of a uniform thin hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center (Fig. 10.9).

Solution Because the hoop is thin, all mass elements dm are the same distance $r = R$ from the axis, and so, applying Equation 10.17, we obtain for the moment of inertia about the z axis through O:

$$I_z = \int r^2\,dm = R^2 \int dm = \boxed{MR^2}$$

Note that this moment of inertia is the same as that of a single particle of mass M located a distance R from the axis of rotation.

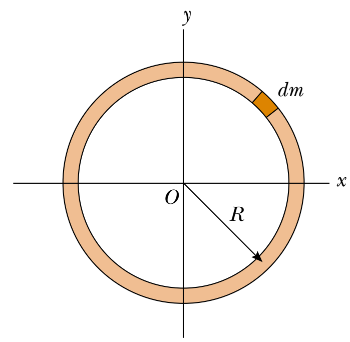


Figure 10.9 (Example 10.5) The mass elements dm of a uniform hoop are all the same distance from O.

Example 10.6 Uniform Rigid Rod

Calculate the moment of inertia of a uniform rigid rod of length L and mass M (Fig. 10.10) about an axis perpendicular to the rod (the y axis) and passing through its center of mass.

Solution The shaded length element dx in Figure 10.10 has a mass dm equal to the mass per unit length λ multiplied by dx:

$$dm = \lambda\,dx = \frac{M}{L}\,dx$$

Substituting this expression for dm into Equation 10.17, with $r^2 = x^2$, we obtain

$$I_y = \int r^2\,dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L}\,dx = \frac{M}{L}\int_{-L/2}^{L/2} x^2\,dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \boxed{\tfrac{1}{12}ML^2}$$

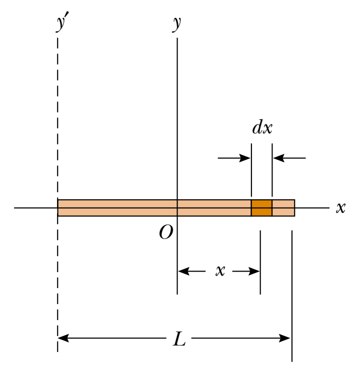


Figure 10.10 (Example 10.6) A uniform rigid rod of length L. The moment of inertia about the y axis is less than that about the y' axis. The latter axis is examined in Example 10.8.

Example 10.7 Uniform Solid Cylinder

A uniform solid cylinder has a radius R, mass M, and length L. Calculate its moment of inertia about its central axis (the z axis in Fig. 10.11).

Solution It is convenient to divide the cylinder into many cylindrical shells, each of which has radius r, thickness dr, and length L, as shown in Figure 10.11. The volume dV of each shell is its cross-sectional area multiplied by its length: $dV = L\,dA = L(2\pi r)\,dr$. If the mass per unit volume is ρ, then the mass of this differential volume element is $dm = \rho\,dV = 2\pi\rho L r\,dr$. Substituting this expression for dm into Equation 10.17, we obtain

$$I_z = \int r^2\,dm = \int r^2(2\pi\rho L r\,dr) = 2\pi\rho L\int_0^R r^3 dr = \tfrac{1}{2}\pi\rho L R^4$$

Because the total volume of the cylinder is $\pi R^2 L$, we see that $\rho = M/V = M/\pi R^2 L$. Substituting this value for ρ into the above result gives

$$\boxed{I_z = \tfrac{1}{2}MR^2}$$

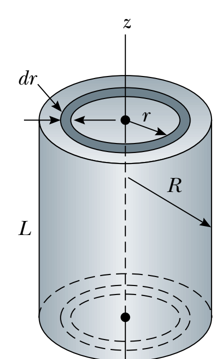


Figure 10.11 (Example 10.7) Calculating I about the z axis for a uniform solid cylinder.

What If? **What if the length of the cylinder in Figure 10.11 is increased to 2*L*, while the mass *M* and radius *R* are held fixed? How does this change the moment of inertia of the cylinder?**

Answer Note that the result for the moment of inertia of a cylinder does not depend on L, the length of the cylinder. In other words, it applies equally well to a long cylinder and a flat disk having the same mass M and radius R. Thus, the moment of inertia of the cylinder would not be affected by changing its length.

Table 10.2 gives the moments of inertia for a number of objects about specific axes. The moments of inertia of rigid objects with simple geometry (high symmetry) are relatively easy to calculate provided the rotation axis coincides with an axis of symmetry. The calculation of moments of inertia about an arbitrary axis can be cumbersome, however, even for a highly symmetric object. Fortunately, use of an important theorem, called the **parallel-axis theorem,** often simplifies the calculation. Suppose the moment of inertia about an axis through the center of mass of an object is I_{CM}. The parallel-axis theorem states that the moment of inertia about any axis parallel to and a distance D away from this axis is

Parallel-axis theorem

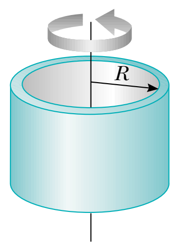
$$I = I_{CM} + MD^2 \tag{10.18}$$

To prove the parallel-axis theorem, suppose that an object rotates in the xy plane about the z axis, as shown in Figure 10.12, and that the coordinates of the center of mass are x_{CM}, y_{CM}. Let the mass element dm have coordinates x, y. Because this

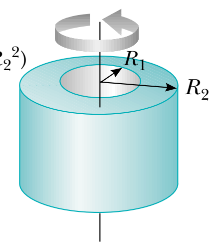
Table 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

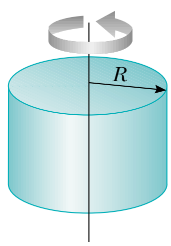
Hoop or thin cylindrical shell
$I_{CM} = MR^2$



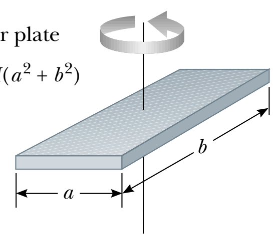
Hollow cylinder
$I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$



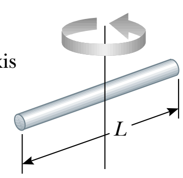
Solid cylinder or disk
$I_{CM} = \frac{1}{2}MR^2$



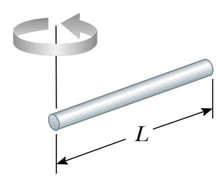
Rectangular plate
$I_{CM} = \frac{1}{12}M(a^2 + b^2)$



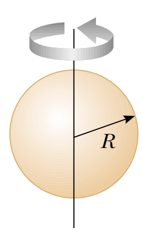
Long thin rod with rotation axis through center
$I_{CM} = \frac{1}{12}ML^2$



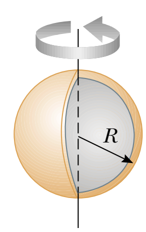
Long thin rod with rotation axis through end
$I = \frac{1}{3}ML^2$

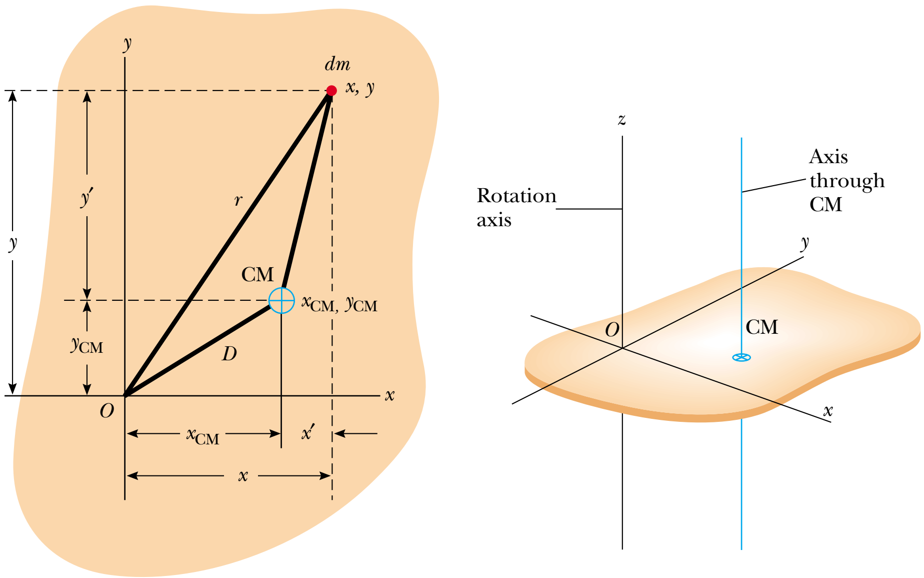


Solid sphere
$I_{CM} = \frac{2}{5}MR^2$



Thin spherical shell
$I_{CM} = \frac{2}{3}MR^2$





(a) (b)

Figure 10.12 (a) The parallel-axis theorem: if the moment of inertia about an axis perpendicular to the figure through the center of mass is I_{CM}, then the moment of inertia about the z axis is $I_z = I_{CM} + MD^2$. (b) Perspective drawing showing the z axis (the axis of rotation) and the parallel axis through the CM.

element is a distance $r = \sqrt{x^2 + y^2}$ from the z axis, the moment of inertia about the z axis is

$$I = \int r^2\,dm = \int (x^2 + y^2)\,dm$$

However, we can relate the coordinates x, y of the mass element dm to the coordinates of this same element located in a coordinate system having the object's center of mass as its origin. If the coordinates of the center of mass are x_{CM}, y_{CM} in the original coordinate system centered on O, then from Figure 10.12a we see that the relationships between the unprimed and primed coordinates are $x = x' + x_{CM}$ and $y = y' + y_{CM}$. Therefore,

$$I = \int [(x' + x_{CM})^2 + (y' + y_{CM})^2]\,dm$$

$$= \int [(x')^2 + (y')^2]\,dm + 2x_{CM} \int x'\,dm + 2y_{CM} \int y'\,dm + (x_{CM}{}^2 + y_{CM}{}^2) \int dm$$

The first integral is, by definition, the moment of inertia about an axis that is parallel to the z axis and passes through the center of mass. The second two integrals are zero because, by definition of the center of mass, $\int x'\,dm = \int y'\,dm = 0$. The last integral is simply MD^2 because $\int dm = M$ and $D^2 = x_{CM}{}^2 + y_{CM}{}^2$. Therefore, we conclude that

$$I = I_{CM} + MD^2$$

Example 10.8 Applying the Parallel-Axis Theorem

Consider once again the uniform rigid rod of mass M and length L shown in Figure 10.10. Find the moment of inertia of the rod about an axis perpendicular to the rod through one end (the y' axis in Fig. 10.10).

Solution Intuitively, we expect the moment of inertia to be greater than $I_{CM} = \frac{1}{12}ML^2$ because there is mass up to a distance of L away from the rotation axis, while the farthest distance in Example 10.6 was only $L/2$. Because the distance between the center-of-mass axis and the y' axis is $D = L/2$, the parallel-axis theorem gives

$$I = I_{CM} + MD^2 = \tfrac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \boxed{\tfrac{1}{3}ML^2}$$

So, it is four times more difficult to change the rotation of a rod spinning about its end than it is to change the motion of one spinning about its center.

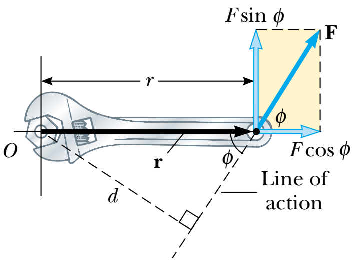


Figure 10.13 The force **F** has a greater rotating tendency about O as F increases and as the moment arm d increases. The component $F \sin \phi$ tends to rotate the wrench about O.

10.6 Torque

Why are a door's hinges and its doorknob placed near opposite edges of the door? Imagine trying to rotate a door by applying a force of magnitude F perpendicular to the door surface but at various distances from the hinges. You will achieve a more rapid rate of rotation for the door by applying the force near the doorknob than by applying it near the hinges.

If you cannot loosen a stubborn bolt with a socket wrench, what would you do in an effort to loosen the bolt? You may intuitively try using a wrench with a longer handle or slip a pipe over the existing wrench to make it longer. This is similar to the situation with the door. You are more successful at causing a change in rotational motion (of the door or the bolt) by applying the force farther away from the rotation axis.

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque** τ (Greek tau). Torque is a vector, but we will consider only its magnitude here and explore its vector nature in Chapter 11.

PITFALL PREVENTION

10.5 Torque Depends on Your Choice of Axis

Like moment of inertia, there is no unique value of the torque—its value depends on your choice of rotation axis.

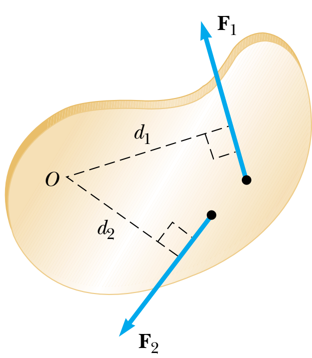
Consider the wrench pivoted on the axis through O in Figure 10.13. The applied force **F** acts at an angle ϕ to the horizontal. We define the magnitude of the torque associated with the force **F** by the expression

$$\tau \equiv rF \sin \phi = Fd \tag{10.19}$$

where r is the distance between the pivot point and the point of application of **F** and d is the perpendicular distance from the pivot point to the line of action of **F**. (The *line of action* of a force is an imaginary line extending out both ends of the vector representing the force. The dashed line extending from the tail of **F** in Figure 10.13 is part of the line of action of **F**.) From the right triangle in Figure 10.13 that has the wrench as its hypotenuse, we see that $d = r \sin \phi$. The quantity d is called the **moment arm** (or *lever arm*) of **F**.

Moment arm

In Figure 10.13, the only component of **F** that tends to cause rotation is $F \sin \phi$, the component perpendicular to a line drawn from the rotation axis to the point of application of the force. The horizontal component $F \cos \phi$, because its line of action passes through O, has no tendency to produce rotation about an axis passing through O. From the definition of torque, we see that the rotating tendency increases as F increases and as d increases. This explains the observation that it is easier to rotate a door if we push at the doorknob rather than at a point close to the hinge. We also want to apply our push as closely perpendicular to the door as we can. Pushing sideways on the doorknob will not cause the door to rotate.



Active Figure 10.14 The force $\mathbf{F}_1$ tends to rotate the object counterclockwise about O, and $\mathbf{F}_2$ tends to rotate it clockwise.

At the Active Figures link at http://www.pse6.com, you can change the magnitudes, directions, and points of application of forces F_1 and F_2 to see how the object accelerates under the action of the two forces.

If two or more forces are acting on a rigid object, as in Figure 10.14, each tends to produce rotation about the axis at O. In this example, $\mathbf{F}_2$ tends to rotate the object clockwise and $\mathbf{F}_1$ tends to rotate it counterclockwise. We use the convention that the sign of the torque resulting from a force is positive if the turning tendency of the force is counterclockwise and is negative if the turning tendency is clockwise. For example, in Figure 10.14, the torque resulting from $\mathbf{F}_1$, which has a moment arm d_1, is positive and equal to $+F_1 d_1$; the torque from $\mathbf{F}_2$ is negative and equal to $-F_2 d_2$. Hence, the *net* torque about O is

$$\sum \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$

Torque should not be confused with force. Forces can cause a change in linear motion, as described by Newton's second law. Forces can also cause a change in rotational motion, but the effectiveness of the forces in causing this change depends on both the forces and the moment arms of the forces, in the combination that we call *torque*. Torque has units of force times length—newton · meters in SI units—and should be reported in these units. Do not confuse torque and work, which have the same units but are very different concepts.

Quick Quiz 10.8 If you are trying to loosen a stubborn screw from a piece of wood with a screwdriver and fail, should you find a screwdriver for which the handle is (a) longer or (b) fatter?

Quick Quiz 10.9 If you are trying to loosen a stubborn bolt from a piece of metal with a wrench and fail, should you find a wrench for which the handle is (a) longer (b) fatter?

Example 10.9 The Net Torque on a Cylinder

A one-piece cylinder is shaped as shown in Figure 10.15, with a core section protruding from the larger drum. The cylinder is free to rotate about the central axis shown in the drawing. A rope wrapped around the drum, which has radius R_1, exerts a force $\mathbf{T}_1$ to the right on the cylinder. A rope wrapped around the core, which has radius R_2, exerts a force $\mathbf{T}_2$ downward on the cylinder.

(A) What is the net torque acting on the cylinder about the rotation axis (which is the z axis in Figure 10.15)?

Solution The torque due to $\mathbf{T}_1$ is $-R_1T_1$. (The sign is negative because the torque tends to produce clockwise rotation.) The torque due to $\mathbf{T}_2$ is $+R_2T_2$. (The sign is positive because the torque tends to produce counterclockwise rotation.) Therefore, the net torque about the rotation axis is

$$\sum \tau = \tau_1 + \tau_2 = \boxed{R_2T_2 - R_1T_1}$$

We can make a quick check by noting that if the two forces are of equal magnitude, the net torque is negative because $R_1 > R_2$. Starting from rest with both forces of equal magnitude acting on it, the cylinder would rotate clockwise because $\mathbf{T}_1$ would be more effective at turning it than would $\mathbf{T}_2$.

(B) Suppose $T_1 = 5.0$ N, $R_1 = 1.0$ m, $T_2 = 15.0$ N, and $R_2 = 0.50$ m. What is the net torque about the rotation axis, and which way does the cylinder rotate starting from rest?

Solution Evaluating the net torque,

$$\sum \tau = (15\text{ N})(0.50\text{ m}) - (5.0\text{ N})(1.0\text{ m}) = \boxed{2.5\text{ N}\cdot\text{m}}$$

Because this torque is positive, the cylinder will begin to rotate in the counterclockwise direction.

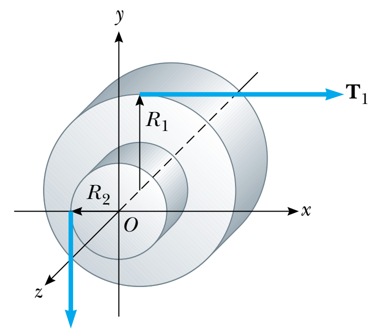


Figure 10.15 (Example 10.9) A solid cylinder pivoted about the z axis through O. The moment arm of $\mathbf{T}_1$ is R_1, and the moment arm of $\mathbf{T}_2$ is R_2.

10.7 Relationship Between Torque and Angular Acceleration

In Chapter 4, we learned that a net force on an object causes an acceleration of the object and that the acceleration is proportional to the net force (Newton's second law). In this section we show the rotational analog of Newton's second law—the angular acceleration of a rigid object rotating about a fixed axis is proportional to the net torque acting about that axis. Before discussing the more complex case of rigid-object rotation, however, it is instructive first to discuss the case of a particle moving in a circular path about some fixed point under the influence of an external force.

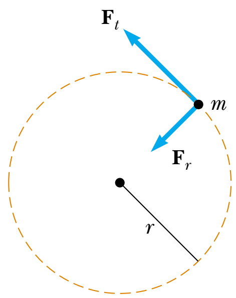


Figure 10.16 A particle rotating in a circle under the influence of a tangential force $\mathbf{F}_t$. A force $\mathbf{F}_r$ in the radial direction also must be present to maintain the circular motion.

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force $\mathbf{F}_t$ and a radial force $\mathbf{F}_r$, as shown in Figure 10.16. The tangential force provides a tangential acceleration $\mathbf{a}_t$, and

$$F_t = ma_t$$

The magnitude of the torque about the center of the circle due to $\mathbf{F}_t$ is

$$\tau = F_t r = (ma_t)r$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$ (see Eq. 10.11), the torque can be expressed as

$$\tau = (mr\alpha)r = (mr^2)\alpha$$

Recall from Equation 10.15 that mr^2 is the moment of inertia of the particle about the z axis passing through the origin, so that

$$\tau = I\alpha \tag{10.20}$$

That is, **the torque acting on the particle is proportional to its angular acceleration,** and the proportionality constant is the moment of inertia. Note that $\tau = I\alpha$ is the rotational analog of Newton's second law of motion, $F = ma$.

Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis, as in Figure 10.17. The object can be regarded as an infinite number of mass elements dm of infinitesimal size. If we impose a Cartesian coordinate system on the object, then each mass element rotates in a circle about the origin, and each has a tangential acceleration $\mathbf{a}_t$ produced by an external tangential force $d\mathbf{F}_t$. For any given element, we know from Newton's second law that

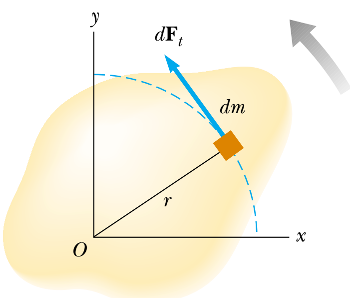


Figure 10.17 A rigid object rotating about an axis through O. Each mass element dm rotates about O with the same angular acceleration α, and the net torque on the object is proportional to α.

$$dF_t = (dm)a_t$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ acts about the origin and is given by

$$d\tau = r\,dF_t = a_t r\,dm$$

Because $a_t = r\alpha$, the expression for $d\tau$ becomes

$$d\tau = \alpha r^2\,dm$$

Although each mass element of the rigid object may have a different linear acceleration $\mathbf{a}_t$, they all have the *same* angular acceleration α. With this in mind, we can integrate the above expression to obtain the net torque $\Sigma\tau$ about O due to the external forces:

$$\sum \tau = \int \alpha r^2\,dm = \alpha \int r^2\,dm$$

where α can be taken outside the integral because it is common to all mass elements. From Equation 10.17, we know that $\int r^2\,dm$ is the moment of inertia of the object about the rotation axis through O, and so the expression for $\Sigma\tau$ becomes

Torque is proportional to angular acceleration

$$\boxed{\sum \tau = I\alpha} \tag{10.21}$$

Note that this is the same relationship we found for a particle moving in a circular path (see Eq. 10.20). So, again we see that the net torque about the rotation axis is proportional to the angular acceleration of the object, with the proportionality factor being I, a quantity that depends upon the axis of rotation and upon the size and shape of the object. In view of the complex nature of the system, the relationship $\Sigma\tau = I\alpha$ is strikingly simple and in complete agreement with experimental observations.

Finally, note that the result $\Sigma\tau = I\alpha$ also applies when the forces acting on the mass elements have radial components as well as tangential components. This is because the line of action of all radial components must pass through the axis of rotation, and hence all radial components produce zero torque about that axis.

Quick Quiz 10.10 You turn off your electric drill and find that the time interval for the rotating bit to come to rest due to frictional torque in the drill is Δt. You replace the bit with a larger one that results in a doubling of the moment of inertia of the entire rotating mechanism of the drill. When this larger bit is rotated at the same angular speed as the first and the drill is turned off, the frictional torque remains the same as that for the previous situation. The time for this second bit to come to rest is (a) $4\Delta t$ (b) $2\Delta t$ (c) Δt (d) $0.5\Delta t$ (e) $0.25\Delta t$ (f) impossible to determine.

Example 10.10 Rotating Rod

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane, as in Figure 10.18. The rod is released from rest in the horizontal position. What is the initial angular acceleration of the rod and the initial linear acceleration of its right end?

Solution We cannot use our kinematic equations to find α or a because the torque exerted on the rod varies with its angular position and so neither acceleration is constant. We have enough information to find the torque, however, which we can then use in Equation 10.21 to find the initial α and then the initial a.

The only force contributing to the torque about an axis through the pivot is the gravitational force $M\mathbf{g}$ exerted on the rod. (The force exerted by the pivot on the rod has zero torque about the pivot because its moment arm is zero.) To compute the torque on the rod, we assume that the gravitational force acts at the center of mass of the rod, as shown in Figure 10.18. The magnitude of the torque due to this force about an axis through the pivot is

$$\tau = Mg\left(\frac{L}{2}\right)$$

With $\Sigma\tau = I\alpha$, and $I = \frac{1}{3}ML^2$ for this axis of rotation (see Table 10.2), we obtain

$$(1) \qquad \alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{\frac{1}{3}ML^2} = \boxed{\frac{3g}{2L}}$$

All points on the rod have this initial angular acceleration.

To find the initial linear acceleration of the right end of the rod, we use the relationship $a_t = r\alpha$ (Eq. 10.11), with $r = L$:

$$a_t = L\alpha = \boxed{\tfrac{3}{2}g}$$

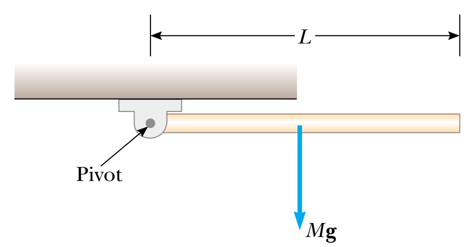


Figure 10.18 (Example 10.10) A rod is free to rotate around a pivot at the left end.

What If? **What if we were to place a penny on the end of the rod and release the rod? Would the penny stay in contact with the rod?**

Answer The result for the initial acceleration of a point on the end of the rod shows that $a_t > g$. A penny will fall at acceleration g. This means that if we place a penny at the end of the rod and then release the rod, the end of the rod falls faster than the penny does! The penny does not stay in contact with the rod. (Try this with a penny and a meter stick!)

This raises the question as to the location on the rod at which we can place a penny that *will* stay in contact as both begin to fall. To find the linear acceleration of an arbitrary point on the rod at a distance $r < L$ from the pivot point, we combine (1) with Equation 10.11:

$$a_t = r\alpha = \frac{3g}{2L}r$$

For the penny to stay in contact with the rod, the limiting case is that the linear acceleration must be equal to that due to gravity:

$$a_t = g = \frac{3g}{2L}r$$

$$r = \tfrac{2}{3}L$$

Thus, a penny placed closer to the pivot than two thirds of the length of the rod will stay in contact with the falling rod while a penny farther out than this point will lose contact.

Conceptual Example 10.11 Falling Smokestacks and Tumbling Blocks

When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground, as shown in Figure 10.19. The same thing happens with a tall tower of children's toy blocks. Why does this happen?

Solution As the smokestack rotates around its base, each higher portion of the smokestack falls with a larger tangential acceleration than the portion below it. (The tangential acceleration of a given point on the smokestack is proportional to the distance of that portion from the base.) As the angular acceleration increases as the smokestack tips farther, higher portions of the smokestack experience an acceleration greater than that which could result from gravity alone; this is similar to the situation described in Example 10.10. This can happen only if these portions are being pulled downward by a force in addition to the gravitational force. The force that causes this to occur is the shear force from lower portions of the smokestack. Eventually the shear force that provides this acceleration is greater than the smokestack can withstand, and the smokestack breaks.

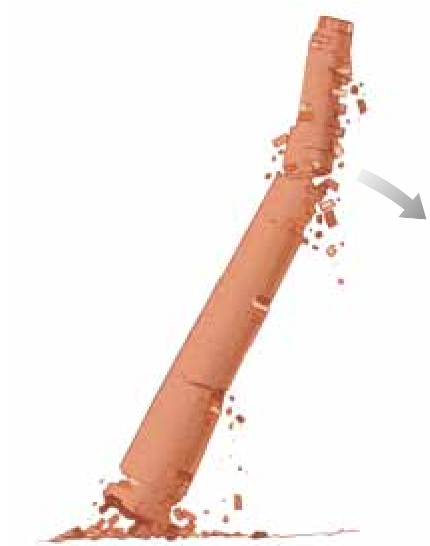


Figure 10.19 (Conceptual Example 10.11) A falling smokestack breaks at some point along its length.

Example 10.12 Angular Acceleration of a Wheel

Interactive

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless horizontal axle, as in Figure 10.20. A light cord wrapped around the wheel supports an object of mass m. Calculate the angular acceleration of the wheel, the linear acceleration of the object, and the tension in the cord.

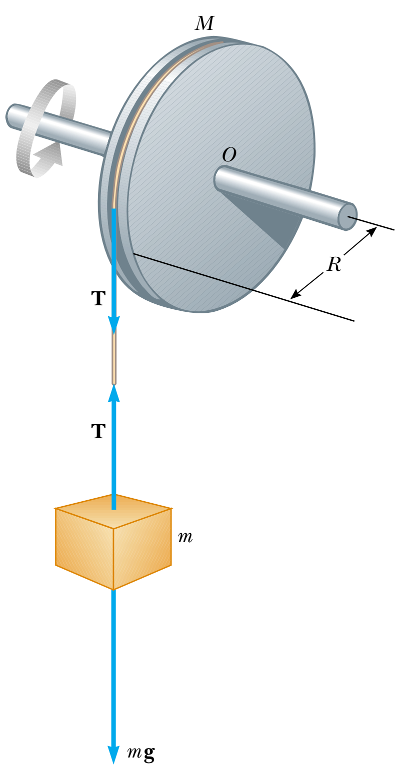


Figure 10.20 (Example 10.12) An object hangs from a cord wrapped around a wheel.

Solution The magnitude of the torque acting on the wheel about its axis of rotation is $\tau = TR$, where T is the force exerted by the cord on the rim of the wheel. (The gravitational force exerted by the Earth on the wheel and the normal force exerted by the axle on the wheel both pass through the axis of rotation and thus produce no torque.) Because $\sum\tau = I\alpha$, we obtain

$$\sum \tau = I\alpha = TR$$

$$(1) \qquad \alpha = \frac{TR}{I}$$

Now let us apply Newton's second law to the motion of the object, taking the downward direction to be positive:

$$\sum F_y = mg - T = ma$$

$$(2) \qquad a = \frac{mg - T}{m}$$

Equations (1) and (2) have three unknowns: α, a, and T. Because the object and wheel are connected by a cord that does not slip, the linear acceleration of the suspended object is equal to the tangential acceleration of a point on the rim of the wheel. Therefore, the angular acceleration α of the wheel and the linear acceleration of the object are related by $a = R\alpha$. Using this fact together with Equations (1) and (2), we obtain

$$(3) \qquad a = R\alpha = \frac{TR^2}{I} = \frac{mg - T}{m}$$

$$(4) \qquad T = \boxed{\frac{mg}{1 + (mR^2/I)}}$$

Substituting Equation (4) into Equation (2) and solving for a and α, we find that

$$(5) \qquad a = \boxed{\frac{g}{1 + (I/mR^2)}}$$

$$\alpha = \frac{a}{R} = \boxed{\frac{g}{R + (I/mR)}}$$

What If? **What if the wheel were to become very massive so that *I* becomes very large? What happens to the acceleration *a* of the object and the tension *T*?**

Answer If the wheel becomes infinitely massive, we can imagine that the object of mass m will simply hang from the cord without causing the wheel to rotate.

We can show this mathematically by taking the limit $I \rightarrow \infty$, so that Equation (5) becomes

$$a = \frac{g}{1 + (I/mR^2)} \longrightarrow 0$$

This agrees with our conceptual conclusion that the object will hang at rest. We also find that Equation (4) becomes

$$T = \frac{mg}{1 + (mR^2/I)} \longrightarrow \frac{mg}{1 + 0} = mg$$

This is consistent with the fact that the object simply hangs at rest in equilibrium between the gravitational force and the tension in the string.

At the Interactive Worked Example link at **http://www.pse6.com,** *you can change the masses of the object and the wheel as well as the radius of the wheel to see the effect on how the system moves.*

Example 10.13 Atwood's Machine Revisited

Interactive

Two blocks having masses m_1 and m_2 are connected to each other by a light cord that passes over two identical frictionless pulleys, each having a moment of inertia I and radius R, as shown in Figure 10.21a. Find the acceleration of each block and the tensions T_1, T_2, and T_3 in the cord. (Assume no slipping between cord and pulleys.)

Solution Compare this situation with the Atwood machine of Example 5.9 (p. 129). The motion of m_1 and m_2 is similar to the motion of the two blocks in that example. The primary differences are that in the present example we have two pulleys and each of the pulleys has mass. Despite these differences, the apparatus in the present example is indeed an Atwood machine.

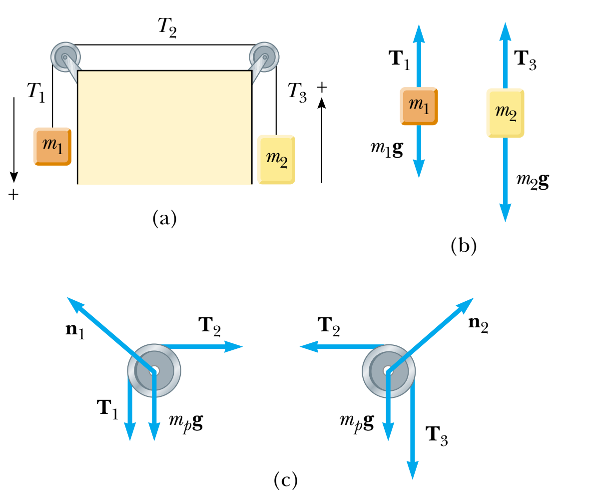


Figure 10.21 (Example 10.13) (a) Another look at Atwood's machine. (b) Free-body diagrams for the blocks. (c) Free-body diagrams for the pulleys, where $m_p\mathbf{g}$ represents the gravitational force acting on each pulley.

We shall define the downward direction as positive for m_1 and upward as the positive direction for m_2. This allows us to represent the acceleration of both masses by a single variable a and also enables us to relate a positive a to a positive (counterclockwise) angular acceleration α of the pulleys. Let us write Newton's second law of motion for each block, using the free-body diagrams for the two blocks as shown in Figure 10.21b:

$$(1) \qquad m_1g - T_1 = m_1a$$

$$(2) \qquad T_3 - m_2g = m_2a$$

Next, we must include the effect of the pulleys on the motion. Free-body diagrams for the pulleys are shown in Figure 10.21c. The net torque about the axle for the pulley on the left is $(T_1 - T_2)R$, while the net torque for the pulley on the right is $(T_2 - T_3)R$. Using the relation $\Sigma\tau = I\alpha$ for each pulley and noting that each pulley has the same angular acceleration α, we obtain

$$(3) \qquad (T_1 - T_2)R = I\alpha$$

$$(4) \qquad (T_2 - T_3)R = I\alpha$$

We now have four equations with five unknowns: α, a, T_1, T_2, and T_3. We also have a fifth equation that relates the accelerations, $a = R\alpha$. These equations can be solved simultaneously. Adding Equations (3) and (4) gives

$$(5) \qquad (T_1 - T_3)R = 2I\alpha$$

Adding Equations (1) and (2) gives

$$T_3 - T_1 + m_1g - m_2g = (m_1 + m_2)a$$

$$(6) \qquad T_1 - T_3 = (m_1 - m_2)g - (m_1 + m_2)a$$

Substituting Equation (6) into Equation (5), we have

$$[(m_1 - m_2)g - (m_1 + m_2)a]R = 2I\alpha$$

Because $\alpha = a/R$, this expression can be simplified to

$$(m_1 - m_2)g - (m_1 + m_2)a = 2I\frac{a}{R^2}$$

$$(7) \qquad a = \boxed{\frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)}}$$

Note that if $m_1 > m_2$, the acceleration is positive; this means that the left block accelerates downward, the right block accelerates upward, and both pulleys accelerate counterclockwise. If $m_1 < m_2$, the acceleration is negative and the motions are reversed. If $m_1 = m_2$, no acceleration occurs at all. You should compare these results with those found in Example 5.9.

The expression for a can be substituted into Equations (1) and (2) to give T_1 and T_3. From Equation (1),

$$\begin{aligned} T_1 &= m_1g - m_1a = m_1(g - a) \\ &= m_1\left(g - \frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)}\right) \\ &= 2m_1g\left(\frac{m_2 + (I/R^2)}{m_1 + m_2 + 2(I/R^2)}\right) \end{aligned}$$

Similarly, from Equation (2),

$$T_3 = m_2g + m_2a = 2m_2g\left(\frac{m_1 + (I/R^2)}{m_1 + m_2 + 2(I/R^2)}\right)$$

Finally, T_2 can be found from Equation (3):

$$\begin{aligned} T_2 &= T_1 - \frac{I\alpha}{R} = T_1 - \frac{Ia}{R^2} \\ &= 2m_1g\left(\frac{m_2 + (I/R^2)}{m_1 + m_2 + 2(I/R^2)}\right) \\ &\quad - \frac{I}{R^2}\left(\frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)}\right) \\ &= \frac{2m_1m_2 + (m_1 + m_2)(I/R^2)}{m_1 + m_2 + 2(I/R^2)}\, g \end{aligned}$$

What If? **What if the pulleys become massless? Does this reduce to a previously solved problem?**

Answer If the pulleys become massless, the system should behave in the same way as the massless-pulley Atwood machine that we investigated in Example 5.9. The only difference is the existence of two pulleys instead of one.

Mathematically, if $I \rightarrow 0$, Equation (7) becomes

$$a = \frac{(m_1 - m_2)g}{m_1 + m_2 + 2(I/R^2)} \longrightarrow a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$$

which is the same result as Equation (3) in Example 5.9. Although the expressions for the three tensions in the present example are different from each other, all three expressions become, in the limit $I \rightarrow 0$,

$$T = \left(\frac{2m_1m_2}{m_1 + m_2}\right)g$$

which is the same as Equation (4) in Example 5.9.

At the Interactive Worked Example link at **http://www.pse6.com,** *you can change the masses of the blocks and the pulleys to see the effect on the motion of the system.*

10.8 Work, Power, and Energy in Rotational Motion

Up to this point in our discussion of rotational motion in this chapter, we focused on an approach involving force, leading to a description of torque on a rigid object. We now see how an energy approach can be useful to us in solving rotational problems.

We begin by considering the relationship between the torque acting on a rigid object and its resulting rotational motion in order to generate expressions for power and a rotational analog to the work–kinetic energy theorem. Consider the rigid object pivoted at O in Figure 10.22. Suppose a single external force $\mathbf{F}$ is applied at P, where $\mathbf{F}$ lies in the plane of the page. The work done by $\mathbf{F}$ on the object as it rotates through an infinitesimal distance $ds = r\,d\theta$ is

$$dW = \mathbf{F}\cdot d\mathbf{s} = (F\sin\phi)r\,d\theta$$

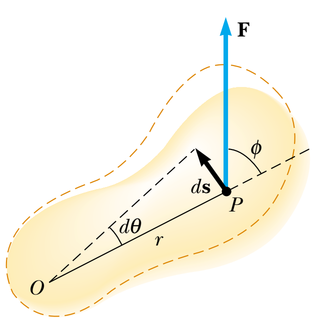


Figure 10.22 A rigid object rotates about an axis through O under the action of an external force $\mathbf{F}$ applied at P.

where $F\sin\phi$ is the tangential component of $\mathbf{F}$, or, in other words, the component of the force along the displacement. Note that *the radial component of* $\mathbf{F}$ *does no work because it is perpendicular to the displacement.*

Because the magnitude of the torque due to **F** about O is defined as $rF \sin \phi$ by Equation 10.19, we can write the work done for the infinitesimal rotation as

$$dW = \tau \, d\theta \tag{10.22}$$

The rate at which work is being done by **F** as the object rotates about the fixed axis through the angle $d\theta$ in a time interval dt is

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

Because dW/dt is the instantaneous power $\mathcal{P}$ (see Section 7.8) delivered by the force and $d\theta/dt = \omega$, this expression reduces to

$$\mathcal{P} = \frac{dW}{dt} = \tau\omega \tag{10.23}$$

Power delivered to a rotating rigid object

This expression is analogous to $\mathcal{P} = Fv$ in the case of linear motion, and the expression $dW = \tau \, d\theta$ is analogous to $dW = F_x \, dx$.

In studying linear motion, we found the energy approach extremely useful in describing the motion of a system. From what we learned of linear motion, we expect that when a symmetric object rotates about a fixed axis, the work done by external forces equals the change in the rotational energy.

To show that this is in fact the case, let us begin with $\Sigma\tau = I\alpha$. Using the chain rule from calculus, we can express the resultant torque as

$$\sum \tau = I\alpha = I \frac{d\omega}{dt} = I \frac{d\omega}{d\theta} \frac{d\theta}{dt} = I \frac{d\omega}{d\theta} \omega$$

Rearranging this expression and noting that $\Sigma\tau \, d\theta = dW$, we obtain

$$\sum \tau \, d\theta = dW = I\omega \, d\omega$$

Integrating this expression, we obtain for the total work done by the net external force acting on a rotating system

$$\sum W = \int_{\omega_i}^{\omega_f} I\omega \, d\omega = \tfrac{1}{2}I\omega_f^{\,2} - \tfrac{1}{2}I\omega_i^{\,2} \tag{10.24}$$

Work–kinetic energy theorem for rotational motion

where the angular speed changes from ω_i to ω_f. That is, the **work–kinetic energy theorem for rotational motion** states that

> the net work done by external forces in rotating a symmetric rigid object about a fixed axis equals the change in the object's rotational energy.

In general, then, combining this with the translational form of the work–kinetic energy theorem from Chapter 7, the net work done by external forces on an object is the change in its *total* kinetic energy, which is the sum of the translational and rotational kinetic energies. For example, when a pitcher throws a baseball, the work done by the pitcher's hands appears as kinetic energy associated with the ball moving through space as well as rotational kinetic energy associated with the spinning of the ball.

In addition to the work–kinetic energy theorem, other energy principles can also be applied to rotational situations. For example, if a system involving rotating objects is isolated, the principle of conservation of energy can be used to analyze the system, as in Example 10.14 below.

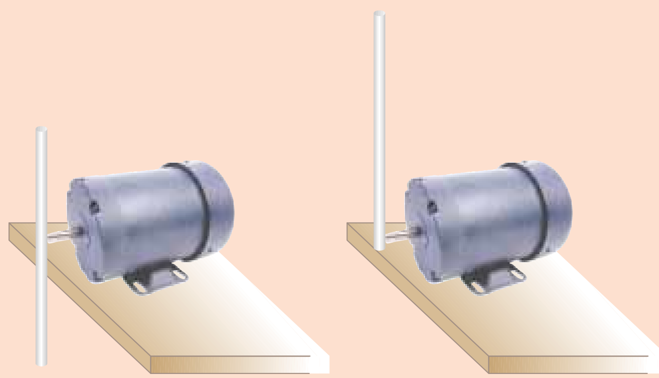
Table 10.3 lists the various equations we have discussed pertaining to rotational motion, together with the analogous expressions for linear motion. The last two equations in Table 10.3, involving angular momentum L, are discussed in Chapter 11 and are included here only for the sake of completeness.

Table 10.3

Useful Equations in Rotational and Linear Motion

Rotational Motion About a Fixed Axis	Linear Motion
Angular speed $\omega = d\theta/dt$	Linear speed $v = dx/dt$
Angular acceleration $\alpha = d\omega/dt$	Linear acceleration $a = dv/dt$
Net torque $\Sigma\tau = I\alpha$	Net force $\Sigma F = ma$
If $\alpha =$ constant $\begin{cases} \omega_f = \omega_i + \alpha_t \\ \theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ \omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \end{cases}$	If $a =$ constant $\begin{cases} v_f = v_i + at \\ x_f = x_i + v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2a(x_f - x_i) \end{cases}$
Work $W = \int_{\theta_i}^{\theta_f} \tau\, d\theta$	Work $W = \int_{x_i}^{x_f} F_x\, dx$
Rotational kinetic energy $K_R = \frac{1}{2}I\omega^2$	Kinetic energy $K = \frac{1}{2}mv^2$
Power $\mathcal{P} = \tau\omega$	Power $\mathcal{P} = Fv$
Angular momentum $L = I\omega$	Linear momentum $p = mv$
Net torque $\Sigma\tau = dL/dt$	Net force $\Sigma F = dp/dt$

Quick Quiz 10.11 A rod is attached to the shaft of a motor at the center of the rod so that the rod is perpendicular to the shaft, as in Figure 10.23a. The motor is turned on and performs work W on the rod, accelerating it to an angular speed ω. The system is brought to rest, and the rod is attached to the shaft of the motor at one end of the rod as in Figure 10.23b. The motor is turned on and performs work W on the rod. The angular speed of the rod in the second situation is (a) 4ω (b) 2ω (c) ω (d) 0.5ω (e) 0.25ω (f) impossible to determine.



(a) (b)

Figure 10.23 (Quick Quiz 10.11) (a) A rod is rotated about its midpoint by a motor. (b) The rod is rotated about one of its ends.

Example 10.14 Rotating Rod Revisited Interactive

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end (Fig 10.24). The rod is released from rest in the horizontal position.

(A) What is its angular speed when it reaches its lowest position?

Solution To conceptualize this problem, consider Figure 10.24 and imagine the rod rotating downward through a quarter turn about the pivot at the left end. In this situation, the angular acceleration of the rod is not constant. Thus, the kinematic equations for rotation (Section 10.2) cannot be used to solve this problem. As we found with translational motion, however, an energy approach can make such a seemingly insoluble problem relatively easy. We categorize this as a conservation of energy problem.

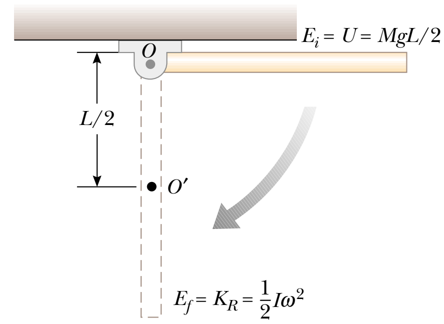


Figure 10.24 (Example 10.14) A uniform rigid rod pivoted at O rotates in a vertical plane under the action of the gravitational force.

To analyze the problem, we consider the mechanical energy of the system of the rod and the Earth. We choose the configuration in which the rod is hanging straight down as the reference configuration for gravitational potential energy and assign a value of zero for this configuration. When the rod is in the horizontal position, it has no rotational kinetic energy. The potential energy of the system in this configuration relative to the reference configuration is $MgL/2$ because the center of mass of the rod is at a height $L/2$ higher than its position in the reference configuration. When the rod reaches its lowest position, the energy is entirely rotational energy $\frac{1}{2}I\omega^2$, where I is the moment of inertia about the pivot, and the potential energy of the system is zero. Because $I = \frac{1}{3}ML^2$ (see Table 10.2) and because the system is isolated with no nonconservative forces acting, we apply conservation of mechanical energy for the system:

$$K_f + U_f = K_i + U_i$$

$$\tfrac{1}{2}I\omega^2 + 0 = \tfrac{1}{2}(\tfrac{1}{3}ML^2)\,\omega^2 = 0 + \tfrac{1}{2}MgL$$

$$\omega = \boxed{\sqrt{\frac{3g}{L}}}$$

(B) Determine the tangential speed of the center of mass and the tangential speed of the lowest point on the rod when it is in the vertical position.

Solution These two values can be determined from the relationship between tangential and angular speeds. We know ω from part (A), and so the tangential speed of the center of mass is

$$v_{\text{CM}} = r\omega = \frac{L}{2}\,\omega = \boxed{\tfrac{1}{2}\sqrt{3gL}}$$

Because r for the lowest point on the rod is twice what it is for the center of mass, the lowest point has a tangential speed v equal to

$$v = 2v_{\text{CM}} = \boxed{\sqrt{3gL}}$$

To finalize this problem, note that the initial configuration in this example is the same as that in Example 10.10. In Example 10.10, however, we could only find the initial angular acceleration of the rod. We cannot use this and the kinematic equations to find the angular speed of the rod at its lowest point because the angular acceleration is not constant. Applying an energy approach in the current example allows us to find something that we cannot in Example 10.10.

At the Interactive Worked Example link at **http://www.pse6.com,** *you can alter the mass and length of the rod and see the effect on the velocity at the lowest point.*

Example 10.15 Energy and the Atwood Machine

Consider two cylinders having different masses m_1 and m_2, connected by a string passing over a pulley, as shown in Figure 10.25. The pulley has a radius R and moment of inertia I about its axis of rotation. The string does not slip on the pulley, and the system is released from rest. Find the linear speeds of the cylinders after cylinder 2 descends through a distance h, and the angular speed of the pulley at this time.

Solution We will solve this problem by applying energy methods to an Atwood machine with a massive pulley. Because the string does not slip, the pulley rotates about the axle. We can neglect friction in the axle because the axle's radius is small relative to that of the pulley, so the frictional torque is much smaller than the torque applied by the two cylinders, provided that their masses are quite different. Consequently, the system consisting of the two cylinders, the pulley, and the Earth is isolated with no nonconservative forces acting; thus, the mechanical energy of the system is conserved.

We define the zero configuration for gravitational potential energy as that which exists when the system is released. From Figure 10.25, we see that the descent of cylinder 2 is associated with a decrease in system potential energy and the rise of cylinder 1 represents an increase in

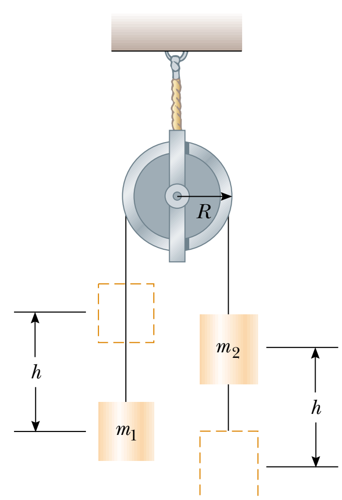


Figure 10.25 (Example 10.15) An Atwood machine.

potential energy. Because $K_i = 0$ (the system is initially at rest), we have

$$K_f + U_f = K_i + U_i$$

$$(\tfrac{1}{2}m_1v_f^2 + \tfrac{1}{2}m_2v_f^2 + \tfrac{1}{2}I\omega_f^2) + (m_1gh - m_2gh) = 0 + 0$$

where v_f is the same for both blocks. Because $v_f = R\omega_f$, this expression becomes

$$\left(\tfrac{1}{2}m_1v_f^2 + \tfrac{1}{2}m_2v_f^2 + \tfrac{1}{2}\frac{I}{R^2}v_f^2\right) = (m_2gh - m_1gh)$$

$$\tfrac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)v_f^2 = (m_2gh - m_1gh)$$

Solving for v_f, we find

$$v_f = \left[\frac{2(m_2 - m_1)gh}{[m_1 + m_2 + (I/R^2)]}\right]^{1/2}$$

The angular speed of the pulley at this instant is

$$\omega_f = \frac{v_f}{R} = \frac{1}{R}\left[\frac{2(m_2 - m_1)gh}{(m_1 + m_2 + (I/R^2))}\right]^{1/2}$$

10.9 Rolling Motion of a Rigid Object

In this section we treat the motion of a rigid object rolling along a flat surface. In general, such motion is very complex. Suppose, for example, that a cylinder is rolling on a straight path such that the axis of rotation remains parallel to its initial orientation in space. As Figure 10.26 shows, a point on the rim of the cylinder moves in a complex path called a *cycloid.* However, we can simplify matters by focusing on the center of mass rather than on a point on the rim of the rolling object. As we see in Figure 10.26, the center of mass moves in a straight line. If an object such as a cylinder rolls without slipping on the surface (we call this *pure rolling motion*), we can show that a simple relationship exists between its rotational and translational motions.

Consider a uniform cylinder of radius R rolling without slipping on a horizontal surface (Fig. 10.27). As the cylinder rotates through an angle θ, its center of mass moves a linear distance $s = R\theta$ (see Eq. 10.1a). Therefore, the linear speed of the center of mass for pure rolling motion is given by

$$v_{\text{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \qquad (10.25)$$

where ω is the angular speed of the cylinder. Equation 10.25 holds whenever a cylinder or sphere rolls without slipping and is the **condition for pure rolling motion.**

PITFALL PREVENTION

10.6 Equation 10.25 Looks Familiar

Equation 10.25 looks very similar to Equation 10.10, so be sure that you are clear on the difference. Equation 10.10 gives the *tangential* speed of a point on a *rotating* object located a distance r from the rotation axis if the object is rotating with angular speed ω. Equation 10.25 gives the *translational* speed of the center of mass of a *rolling* object of radius R rotating with angular speed ω.

Henry Leap and Jim Lehman

Figure 10.26 One light source at the center of a rolling cylinder and another at one point on the rim illustrate the different paths these two points take. The center moves in a straight line (green line), while the point on the rim moves in the path called a cycloid (red curve).

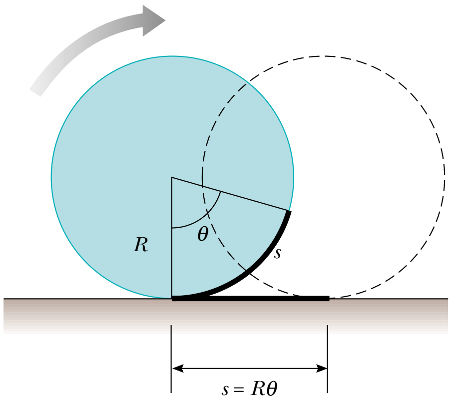


Figure 10.27 For pure rolling motion, as the cylinder rotates through an angle θ, its center moves a linear distance $s = R\theta$.

The magnitude of the linear acceleration of the center of mass for pure rolling motion is

$$a_{\mathrm{CM}} = \frac{dv_{\mathrm{CM}}}{dt} = R\frac{d\omega}{dt} = R\alpha \tag{10.26}$$

where α is the angular acceleration of the cylinder.

The linear velocities of the center of mass and of various points on and within the cylinder are illustrated in Figure 10.28. A short time after the moment shown in the drawing, the rim point labeled P might rotate from the six o'clock position to, say, the seven o'clock position, while the point Q would rotate from the ten o'clock position to the eleven o'clock position, and so on. Note that the linear velocity of any point is in a direction perpendicular to the line from that point to the contact point P. At any instant, the part of the rim that is at point P is at rest relative to the surface because slipping does not occur.

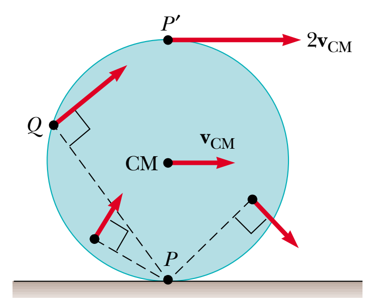


Figure 10.28 All points on a rolling object move in a direction perpendicular to an axis through the instantaneous point of contact P. In other words, all points rotate about P. The center of mass of the object moves with a velocity $\mathbf{v}_{\mathrm{CM}}$, and the point P' moves with a velocity $2\mathbf{v}_{\mathrm{CM}}$.

All points on the cylinder have the same angular speed. Therefore, because the distance from P' to P is twice the distance from P to the center of mass, P' has a speed $2v_{\mathrm{CM}} = 2R\omega$. To see why this is so, let us model the rolling motion of the cylinder in Figure 10.29 as a combination of translational (linear) motion and rotational motion. For the pure translational motion shown in Figure 10.29a, imagine that the cylinder does not rotate, so that each point on it moves to the right with speed v_{CM}. For the pure rotational motion shown in Figure 10.29b, imagine that a rotation axis through the center of mass is stationary, so that each point on the cylinder has the same angular speed ω. The combination of these two motions represents the rolling motion shown in Figure 10.29c. Note in Figure 10.29c that the top of the cylinder has linear speed $v_{\mathrm{CM}} + R\omega = v_{\mathrm{CM}} + v_{\mathrm{CM}} = 2v_{\mathrm{CM}}$, which is greater than the linear speed of any other point on the cylinder. As mentioned earlier, the center of mass moves with linear speed v_{CM} while the contact point between the surface and cylinder has a linear speed of zero.

We can express the total kinetic energy of the rolling cylinder as

$$K = \tfrac{1}{2}I_P\omega^2 \tag{10.27}$$

where I_P is the moment of inertia about a rotation axis through P. Applying the parallel-axis theorem, we can substitute $I_P = I_{\mathrm{CM}} + MR^2$ into Equation 10.27 to obtain

$$K = \tfrac{1}{2}I_{\mathrm{CM}}\omega^2 + \tfrac{1}{2}MR^2\omega^2$$

or, because $v_{\mathrm{CM}} = R\omega$,

$$\boxed{K = \tfrac{1}{2}I_{\mathrm{CM}}\omega^2 + \tfrac{1}{2}Mv_{\mathrm{CM}}^2} \tag{10.28}$$

Total kinetic energy of a rolling object

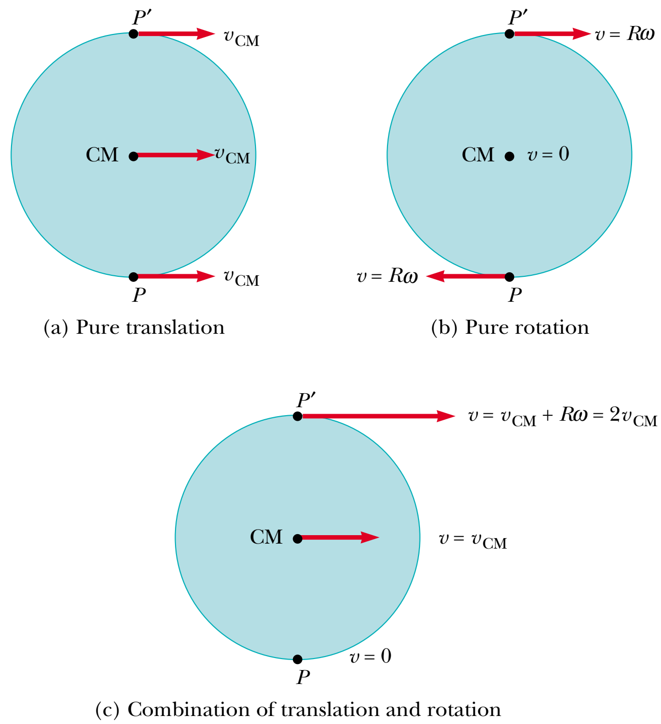
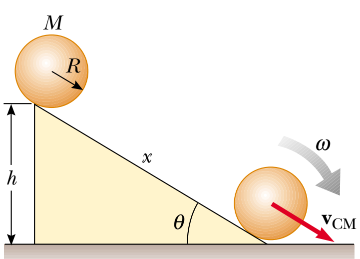


Figure 10.29 The motion of a rolling object can be modeled as a combination of pure translation and pure rotation.

The term $\frac{1}{2}I_{CM}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass, and the term $\frac{1}{2}Mv_{CM}{}^2$ represents the kinetic energy the cylinder would have if it were just translating through space without rotating. Thus, we can say that the **total kinetic energy of a rolling object is the sum of the rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.**

We can use energy methods to treat a class of problems concerning the rolling motion of an object down a rough incline. For example, consider Figure 10.30, which shows a sphere rolling without slipping after being released from rest at the top of the incline. Note that accelerated rolling motion is possible only if a friction force is present between the sphere and the incline to produce a net torque about the center of mass. Despite the presence of friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant. (On the other hand, if the sphere were to slip, mechanical energy of the sphere–incline–Earth system would be lost due to the nonconservative force of kinetic friction.)



Active Figure 10.30 A sphere rolling down an incline. Mechanical energy of the sphere–incline–Earth system is conserved if no slipping occurs.

At the Active Figures link at http://www.pse6.com, you can roll several objects down the hill and see how the final speed depends on the type of object.

Using the fact that $v_{CM} = R\omega$ for pure rolling motion, we can express Equation 10.28 as

$$K = \frac{1}{2}I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \frac{1}{2}Mv_{CM}{}^2$$

$$K = \frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}{}^2 \tag{10.29}$$

For the system of the sphere and the Earth, we define the zero configuration of gravitational potential energy to be when the sphere is at the bottom of the incline. Thus, conservation of mechanical energy gives us

$$K_f + U_f = K_i + U_i$$

$$\frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}{}^2 + 0 = 0 + Mgh$$

$$v_{CM} = \left(\frac{2gh}{1 + (I_{CM}/MR^2)}\right)^{1/2} \tag{10.30}$$

Quick Quiz 10.12 A ball rolls without slipping down incline A, starting from rest. At the same time, a box starts from rest and slides down incline B, which is identical to incline A except that it is frictionless. Which arrives at the bottom first? (a) the ball (b) the box (c) Both arrive at the same time. (d) impossible to determine

Quick Quiz 10.13 Two solid spheres roll down an incline, starting from rest. Sphere A has twice the mass and twice the radius of sphere B. Which arrives at the bottom first? (a) sphere A (b) sphere B (c) Both arrive at the same time. (d) impossible to determine

Quick Quiz 10.14 Two spheres roll down an incline, starting from rest. Sphere A has the same mass and radius as sphere B, but sphere A is solid while sphere B is hollow. Which arrives at the bottom first? (a) sphere A (b) sphere B (c) Both arrive at the same time. (d) impossible to determine

Example 10.16 Sphere Rolling Down an Incline

For the solid sphere shown in Figure 10.30, calculate the linear speed of the center of mass at the bottom of the incline and the magnitude of the linear acceleration of the center of mass.

Solution For a uniform solid sphere, $I_{\text{CM}} = \frac{2}{5}MR^2$ (see Table 10.2), and therefore Equation 10.30 gives

$$v_{\text{CM}} = \left(\frac{2gh}{1 + (\frac{2}{5}MR^2/MR^2)}\right)^{1/2} = \boxed{(\tfrac{10}{7}gh)^{1/2}}$$

Notice that this is less than $\sqrt{2gh}$, which is the speed an object would have if it simply slid down the incline without rotating (see Example 8.7).

To calculate the linear acceleration of the center of mass, we note that the vertical displacement is related to the displacement x along the incline through the relationship $h = x \sin\theta$. Hence, after squaring both sides, we can express the equation above as

$$v_{\text{CM}}^2 = \tfrac{10}{7}gx\sin\theta$$

Comparing this with the expression from kinematics, $v_{\text{CM}}^2 = 2a_{\text{CM}}x$ (see Eq. 2.13), we see that the acceleration of the center of mass is

$$a_{\text{CM}} = \tfrac{5}{7}g\sin\theta$$

These results are interesting because both the speed and the acceleration of the center of mass are *independent* of the mass and the radius of the sphere! That is, **all homogeneous solid spheres experience the same speed and acceleration on a given incline,** as we argued in the answer to Quick Quiz 10.13.

If we were to repeat the acceleration calculation for a hollow sphere, a solid cylinder, or a hoop, we would obtain similar results in which only the factor in front of $g\sin\theta$ would differ. The constant factors that appear in the expressions for v_{CM} and a_{CM} depend only on the moment of inertia about the center of mass for the specific object. In all cases, the acceleration of the center of mass is *less* than $g\sin\theta$, the value the acceleration would have if the incline were frictionless and no rolling occurred.

SUMMARY

If a particle moves in a circular path of radius r through an angle θ (measured in radians), the arc length it moves through is $s = r\theta$.

The **angular position** of a rigid object is defined as the angle θ between a reference line attached to the object and a reference line fixed in space. The **angular displacement** of a particle moving in a circular path or a rigid object rotating about a fixed axis is $\Delta\theta \equiv \theta_f - \theta_i$.

Take a practice test for this chapter by clicking on the Practice Test link at http://www.pse6.com.

The **instantaneous angular speed** of a particle moving in a circular path or of a rigid object rotating about a fixed axis is

$$\omega \equiv \frac{d\theta}{dt} \tag{10.3}$$

The **instantaneous angular acceleration** of a particle moving in a circular path or a rotating rigid object is

$$\alpha \equiv \frac{d\omega}{dt} \tag{10.5}$$

When a rigid object rotates about a fixed axis, every part of the object has the same angular speed and the same angular acceleration.

If an object rotates about a fixed axis under constant angular acceleration, one can apply equations of kinematics that are analogous to those for linear motion under constant linear acceleration:

$$\omega_f = \omega_i + \alpha t \tag{10.6}$$

$$\theta_f = \theta_i + \omega_i t + \tfrac{1}{2}\alpha t^2 \tag{10.7}$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \tag{10.8}$$

$$\theta_f = \theta_i + \tfrac{1}{2}(\omega_i + \omega_f)\, t \tag{10.9}$$

A useful technique in solving problems dealing with rotation is to visualize a linear version of the same problem.

When a rigid object rotates about a fixed axis, the angular position, angular speed, and angular acceleration are related to the linear position, linear speed, and linear acceleration through the relationships

$$s = r\theta \tag{10.1a}$$

$$v = r\omega \tag{10.10}$$

$$a_t = r\alpha \tag{10.11}$$

The **moment of inertia of a system of particles** is defined as

$$I \equiv \sum_i m_i r_i^2 \tag{10.15}$$

If a rigid object rotates about a fixed axis with angular speed ω, its **rotational kinetic energy** can be written

$$K_R = \tfrac{1}{2} I\omega^2 \tag{10.16}$$

where I is the moment of inertia about the axis of rotation.

The **moment of inertia of a rigid object** is

$$I = \int r^2\, dm \tag{10.17}$$

where r is the distance from the mass element dm to the axis of rotation.

The magnitude of the **torque** associated with a force **F** acting on an object is

$$\tau = Fd \tag{10.19}$$

where d is the moment arm of the force, which is the perpendicular distance from the rotation axis to the line of action of the force. Torque is a measure of the tendency of the force to change the rotation of the object about some axis.

If a rigid object free to rotate about a fixed axis has a **net external torque** acting on it, the object undergoes an angular acceleration α, where

$$\sum \tau = I\alpha \tag{10.21}$$

The rate at which work is done by an external force in rotating a rigid object about a fixed axis, or the **power** delivered, is

$$\mathcal{P} = \tau\omega \tag{10.23}$$

If work is done on a rigid object and the only result of the work is rotation about a fixed axis, the net work done by external forces in rotating the object equals the change in the rotational kinetic energy of the object:

$$\sum W = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2 \tag{10.24}$$

The **total kinetic energy** of a rigid object rolling on a rough surface without slipping equals the rotational kinetic energy about its center of mass, $\frac{1}{2}I_{\text{CM}}\omega^2$, plus the translational kinetic energy of the center of mass, $\frac{1}{2}Mv_{\text{CM}}^2$:

$$K = \tfrac{1}{2}I_{\text{CM}}\omega^2 + \tfrac{1}{2}Mv_{\text{CM}}^2 \tag{10.28}$$

QUESTIONS

1. What is the angular speed of the second hand of a clock? What is the direction of $\boldsymbol{\omega}$ as you view a clock hanging on a vertical wall? What is the magnitude of the angular acceleration vector $\boldsymbol{\alpha}$ of the second hand?
2. One blade of a pair of scissors rotates counterclockwise in the xy plane. What is the direction of $\boldsymbol{\omega}$? What is the direction of $\boldsymbol{\alpha}$ if the magnitude of the angular velocity is decreasing in time?
3. Are the kinematic expressions for θ, ω, and α valid when the angular position is measured in degrees instead of in radians?
4. If a car's standard tires are replaced with tires of larger outside diameter, will the reading of the speedometer change? Explain.
5. Suppose $a = b$ and $M > m$ for the system of particles described in Figure 10.8. About which axis (x, y, or z) does the moment of inertia have the smallest value? the largest value?
6. Suppose that the rod in Figure 10.10 has a nonuniform mass distribution. In general, would the moment of inertia about the y axis still be equal to $ML^2/12$? If not, could the moment of inertia be calculated without knowledge of the manner in which the mass is distributed?
7. Suppose that just two external forces act on a stationary rigid object and the two forces are equal in magnitude and opposite in direction. Under what condition does the object start to rotate?
8. Suppose a pencil is balanced on a perfectly frictionless table. If it falls over, what is the path followed by the center of mass of the pencil?
9. Explain how you might use the apparatus described in Example 10.12 to determine the moment of inertia of the wheel. (If the wheel does not have a uniform mass density, the moment of inertia is not necessarily equal to $\frac{1}{2}MR^2$.)
10. Using the results from Example 10.12, how would you calculate the angular speed of the wheel and the linear speed of the suspended counterweight at $t = 2$ s, if the system is released from rest at $t = 0$? Is the expression $v = R\omega$ valid in this situation?
11. If a small sphere of mass M were placed at the end of the rod in Figure 10.24, would the result for ω be greater than, less than, or equal to the value obtained in Example 10.14?
12. Explain why changing the axis of rotation of an object changes its moment of inertia.
13. The moment of inertia of an object depends on the choice of rotation axis, as suggested by the parallel-axis theorem. Argue that an axis passing through the center of mass of an object must be the axis with the smallest moment of inertia.
14. Suppose you remove two eggs from the refrigerator, one hard-boiled and the other uncooked. You wish to determine which is the hard-boiled egg without breaking the eggs. This can be done by spinning the two eggs on the floor and comparing the rotational motions. Which egg spins faster? Which rotates more uniformly? Explain.
15. Which of the entries in Table 10.2 applies to finding the moment of inertia of a long straight sewer pipe rotating about its axis of symmetry? Of an embroidery hoop rotating about an axis through its center and perpendicular to its plane? Of a uniform door turning on its hinges? Of a coin turning about an axis through its center and perpendicular to its faces?
16. Is it possible to change the translational kinetic energy of an object without changing its rotational energy?
17. Must an object be rotating to have a nonzero moment of inertia?
18. If you see an object rotating, is there necessarily a net torque acting on it?
19. Can a (momentarily) stationary object have a nonzero angular acceleration?
20. In a tape recorder, the tape is pulled past the read-and-write heads at a constant speed by the drive mechanism. Consider the reel from which the tape is pulled. As the tape is pulled from it, the radius of the roll of remaining tape decreases. How does the torque on the reel change with time? How does the angular speed of the reel change in time? If the drive mechanism is switched on so that the

tape is suddenly jerked with a large force, is the tape more likely to break when it is being pulled from a nearly full reel or from a nearly empty reel?

21. The polar diameter of the Earth is slightly less than the equatorial diameter. How would the moment of inertia of the Earth about its axis of rotation change if some mass from near the equator were removed and transferred to the polar regions to make the Earth a perfect sphere?

22. Suppose you set your textbook sliding across a gymnasium floor with a certain initial speed. It quickly stops moving because of a friction force exerted on it by the floor. Next, you start a basketball rolling with the same initial speed. It keeps rolling from one end of the gym to the other. Why does the basketball roll so far? Does friction significantly affect its motion?

23. When a cylinder rolls on a horizontal surface as in Figure 10.28, do any points on the cylinder have only a vertical component of velocity at some instant? If so, where are they?

24. Three objects of uniform density—a solid sphere, a solid cylinder, and a hollow cylinder—are placed at the top of an incline (Fig. Q10.24). They are all released from rest at the same elevation and roll without slipping. Which object reaches the bottom first? Which reaches it last? Try this at home and note that the result is independent of the masses and the radii of the objects.

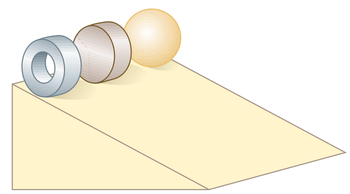


Figure Q10.24 Which object wins the race?

25. In a soap-box derby race, the cars have no engines; they simply coast down a hill to race with one another. Suppose you are designing a car for a coasting race. Do you want to use large wheels or small wheels? Do you want to use solid disk-like wheels or hoop-like wheels? Should the wheels be heavy or light?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*

= coached solution with hints available at http://www.pse6.com = computer useful in solving problem

= paired numerical and symbolic problems

Section 10.1 Angular Position, Velocity, and Acceleration

1. During a certain period of time, the angular position of a swinging door is described by $\theta = 5.00 + 10.0t + 2.00t^2$, where θ is in radians and t is in seconds. Determine the angular position, angular speed, and angular acceleration of the door (a) at $t = 0$ (b) at $t = 3.00$ s.

Section 10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

2. A dentist's drill starts from rest. After 3.20 s of constant angular acceleration, it turns at a rate of 2.51×10^4 rev/min. (a) Find the drill's angular acceleration. (b) Determine the angle (in radians) through which the drill rotates during this period.

3. A wheel starts from rest and rotates with constant angular acceleration to reach an angular speed of 12.0 rad/s in 3.00 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time.

4. An airliner arrives at the terminal, and the engines are shut off. The rotor of one of the engines has an initial clockwise angular speed of 2 000 rad/s. The engine's rotation slows with an angular acceleration of magnitude 80.0 rad/s^2. (a) Determine the angular speed after 10.0 s. (b) How long does it take the rotor to come to rest?

5. An electric motor rotating a grinding wheel at 100 rev/min is switched off. With constant negative angular acceleration of magnitude 2.00 rad/s^2, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn while it is slowing down?

6. A centrifuge in a medical laboratory rotates at an angular speed of 3 600 rev/min. When switched off, it rotates 50.0 times before coming to rest. Find the constant angular acceleration of the centrifuge.

7. The tub of a washer goes into its spin cycle, starting from rest and gaining angular speed steadily for 8.00 s, at which time it is turning at 5.00 rev/s. At this point the person doing the laundry opens the lid, and a safety switch turns off the washer. The tub smoothly slows to rest in 12.0 s. Through how many revolutions does the tub turn while it is in motion?

8. A rotating wheel requires 3.00 s to rotate through 37.0 revolutions. Its angular speed at the end of the 3.00-s interval is 98.0 rad/s. What is the constant angular acceleration of the wheel?

9. (a) Find the angular speed of the Earth's rotation on its axis. As the Earth turns toward the east, we see the sky turning toward the west at this same rate.
(b) *The rainy Pleiads wester*
And seek beyond the sea
The head that I shall dream of
That shall not dream of me.
–A. E. Housman (© Robert E. Symons)

Cambridge, England, is at longitude 0°, and Saskatoon, Saskatchewan, is at longitude 107° west. How much time elapses after the Pleiades set in Cambridge until these stars fall below the western horizon in Saskatoon?

10. A merry-go-round is stationary. A dog is running on the ground just outside its circumference, moving with a constant angular speed of 0.750 rad/s. The dog does not change his pace when he sees what he has been looking for: a bone resting on the edge of the merry-go-round one third of a revolution in front of him. At the instant the dog sees the bone ($t = 0$), the merry-go-round begins to move in the direction the dog is running, with a constant angular acceleration of 0.015 0 rad/s^2. (a) At what time will the dog reach the bone? (b) The confused dog keeps running and passes the bone. How long after the merry-go-round starts to turn do the dog and the bone draw even with each other for the second time?

Section 10.3 Angular and Linear Quantities

11. Make an order-of-magnitude estimate of the number of revolutions through which a typical automobile tire turns in 1 yr. State the quantities you measure or estimate and their values.

12. A racing car travels on a circular track of radius 250 m. If the car moves with a constant linear speed of 45.0 m/s, find (a) its angular speed and (b) the magnitude and direction of its acceleration.

13. A wheel 2.00 m in diameter lies in a vertical plane and rotates with a constant angular acceleration of 4.00 rad/s^2. The wheel starts at rest at $t = 0$, and the radius vector of a certain point P on the rim makes an angle of 57.3° with the horizontal at this time. At $t = 2.00$ s, find (a) the angular speed of the wheel, (b) the tangential speed and the total acceleration of the point P, and (c) the angular position of the point P.

14. Figure P10.14 shows the drive train of a bicycle that has wheels 67.3 cm in diameter and pedal cranks 17.5 cm long. The cyclist pedals at a steady angular rate of 76.0 rev/min. The chain engages with a front sprocket 15.2 cm in diameter and a rear sprocket 7.00 cm in diameter. (a) Calculate the speed of a link of the chain relative to the bicycle frame. (b) Calculate the angular speed of the bicycle wheels. (c) Calculate the speed of the bicycle relative to the road. (d) What pieces of data, if any, are not necessary for the calculations?

Figure P10.14

15. A discus thrower (Fig. P10.15) accelerates a discus from rest to a speed of 25.0 m/s by whirling it through 1.25 rev. Assume the discus moves on the arc of a circle 1.00 m in radius. (a) Calculate the final angular speed of the discus. (b) Determine the magnitude of the angular acceleration of the discus, assuming it to be constant. (c) Calculate the time interval required for the discus to accelerate from rest to 25.0 m/s.

Bruce Ayers/Stone/Getty

Figure P10.15

16. A car accelerates uniformly from rest and reaches a speed of 22.0 m/s in 9.00 s. If the diameter of a tire is 58.0 cm, find (a) the number of revolutions the tire makes during this motion, assuming that no slipping occurs. (b) What is the final angular speed of a tire in revolutions per second?

17. A disk 8.00 cm in radius rotates at a constant rate of 1 200 rev/min about its central axis. Determine (a) its angular speed, (b) the tangential speed at a point 3.00 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2.00 s.

18. A car traveling on a flat (unbanked) circular track accelerates uniformly from rest with a tangential acceleration of 1.70 m/s^2. The car makes it one quarter of the way around the circle before it skids off the track. Determine the coefficient of static friction between the car and track from these data.

19. Consider a tall building located on the Earth's equator. As the Earth rotates, a person on the top floor of the building moves faster than someone on the ground with respect to an inertial reference frame, because the latter person is closer to the Earth's axis. Consequently, if an object is dropped from the top floor to the ground a distance h below, it lands east of the point vertically below where it was dropped. (a) How far to the east will the object land? Express your answer in terms of h, g, and the angular speed ω of the Earth. Neglect air resistance, and assume that the free-fall acceleration is constant over this range of heights. (b) Evaluate the eastward displacement for $h = 50.0$ m. (c) In your judgment, were we justified in ignoring this aspect of the *Coriolis effect* in our previous study of free fall?

Section 10.4 Rotational Kinetic Energy

20. Rigid rods of negligible mass lying along the y axis connect three particles (Fig. P10.20). If the system rotates about the x axis with an angular speed of 2.00 rad/s, find (a) the moment of inertia about the x axis and the total rotational kinetic energy evaluated from $\frac{1}{2}I\omega^2$ and (b) the tangential speed of each particle and the total kinetic energy evaluated from $\Sigma\frac{1}{2}m_iv_i^2$.

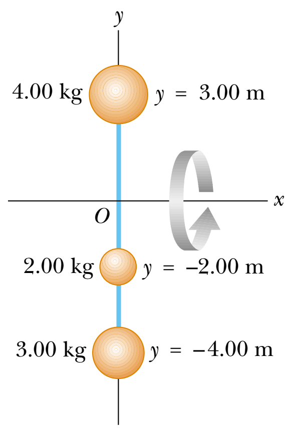


Figure P10.20

21. The four particles in Figure P10.21 are connected by rigid rods of negligible mass. The origin is at the center of the rectangle. If the system rotates in the xy plane about the z axis with an angular speed of 6.00 rad/s, calculate (a) the moment of inertia of the system about the z axis and (b) the rotational kinetic energy of the system.

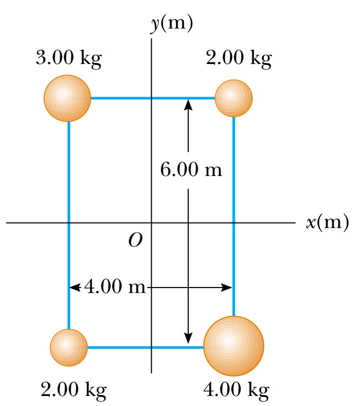


Figure P10.21

22. Two balls with masses M and m are connected by a rigid rod of length L and negligible mass as in Figure P10.22. For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axis passes through the center of mass. Show that this moment of inertia is $I = \mu L^2$, where $\mu = mM/(m + M)$.

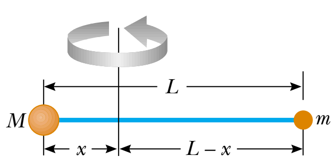


Figure P10.22

Section 10.5 Calculation of Moments of Inertia

23. Three identical thin rods, each of length L and mass m, are welded perpendicular to one another as shown in Figure P10.23. The assembly is rotated about an axis that passes through the end of one rod and is parallel to another. Determine the moment of inertia of this structure.

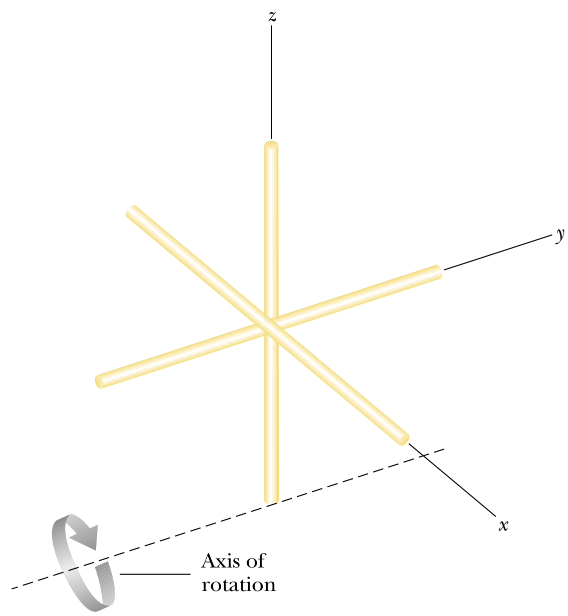


Figure P10.23

24. Figure P10.24 shows a side view of a car tire. Model it as having two sidewalls of uniform thickness 0.635 cm and a tread wall of uniform thickness 2.50 cm and width 20.0 cm. Assume the rubber has uniform density 1.10×10^3 kg/m^3. Find its moment of inertia about an axis through its center.

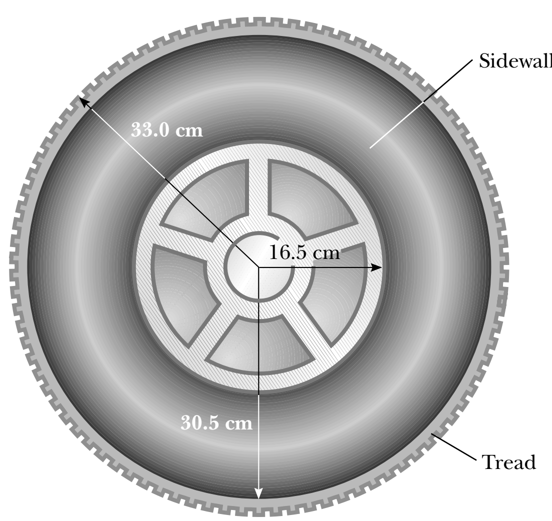


Figure P10.24

25. A uniform thin solid door has height 2.20 m, width 0.870 m, and mass 23.0 kg. Find its moment of inertia for rotation on its hinges. Is any piece of data unnecessary?

26. *Attention! About face!* Compute an order-of-magnitude estimate for the moment of inertia of your body as you stand tall and turn about a vertical axis through the top of your head and the point halfway between your ankles. In your solution state the quantities you measure or estimate and their values.

27. The density of the Earth, at any distance r from its center, is approximately

$$\rho = [14.2 - 11.6(r/R)] \times 10^3 \text{ kg/m}^3$$

where R is the radius of the Earth. Show that this density leads to a moment of inertia $I = 0.330MR^2$ about an axis through the center, where M is the mass of the Earth.

28. Calculate the moment of inertia of a thin plate, in the shape of a right triangle, about an axis that passes through one end of the hypotenuse and is parallel to the opposite leg of the triangle, as in Figure P10.28a. Let M represent the mass of the triangle and L the length of the base of the triangle perpendicular to the axis of rotation. Let h represent the height of the triangle and w the thickness of the plate, much smaller than L or h. Do the calculation in either or both of the following ways, as your instructor assigns:

(a) Use Equation 10.17. Let an element of mass consist of a vertical ribbon within the triangle, of width dx, height y, and thickness w. With x representing the location of the ribbon, show that $y = hx/L$. Show that the density of the material is given by $\rho = 2M/Lwh$. Show that the mass of the ribbon is $dm = \rho yw\, dx = 2Mx\, dx/L^2$. Proceed to use Equation 10.17 to calculate the moment of inertia.

(b) Let I represent the unknown moment of inertia about an axis through the corner of the triangle. Note that Example 9.15 demonstrates that the center of mass of the triangle is two thirds of the way along the length L, from the corner toward the side of height h. Let I_{CM} represent the moment of inertia of the triangle about an axis through the center of mass and parallel to side h. Demonstrate that $I = I_{CM} + 4ML^2/9$. Figure P10.28b shows the same object in a different orientation. Demonstrate that the moment of inertia of the triangular plate, about the y axis is $I_h = I_{CM} + ML^2/9$. Demonstrate that the sum of the moments of inertia of the triangles shown in parts (a) and (b) of the figure must be the moment of inertia of a rectangular sheet of mass $2M$ and length L, rotating like a door about an axis along its edge of height h. Use information in Table 10.2 to write down the moment of inertia of the rectangle, and set it equal to the sum of the moments of inertia of the two triangles. Solve the equation to find the moment of inertia of a triangle about an axis through its center of mass, in terms of M and L. Proceed to find the original unknown I.

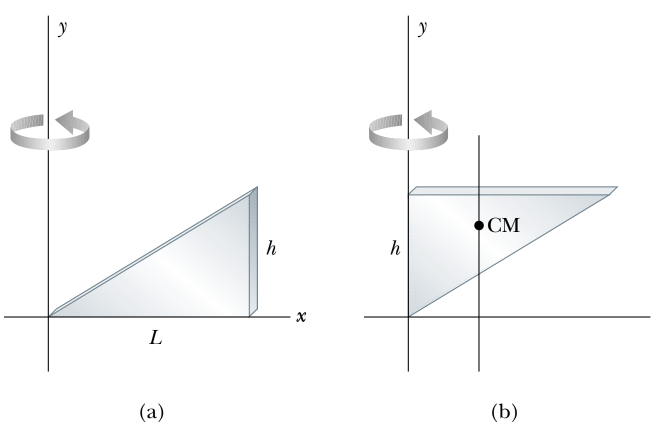


Figure P10.28

29. Many machines employ cams for various purposes, such as opening and closing valves. In Figure P10.29, the cam is a circular disk rotating on a shaft that does not pass through the center of the disk. In the manufacture of the cam, a uniform solid cylinder of radius R is first machined. Then an off-center hole of radius $R/2$ is drilled, parallel to the axis of the cylinder, and centered at a point a distance $R/2$ from the center of the cylinder. The cam, of mass M, is then slipped onto the circular shaft and welded into place. What is the kinetic energy of the cam when it is rotating with angular speed ω about the axis of the shaft?

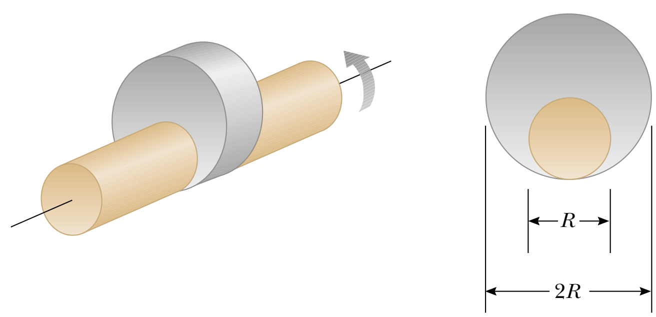


Figure P10.29

Section 10.6 Torque

30. The fishing pole in Figure P10.30 makes an angle of 20.0° with the horizontal. What is the torque exerted by the fish about an axis perpendicular to the page and passing through the fisher's hand?

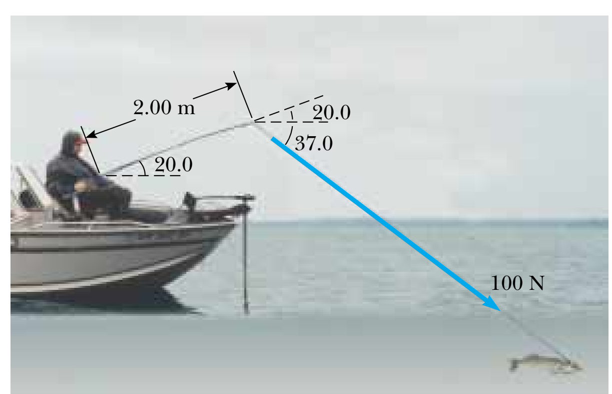


Figure P10.30

31. Find the net torque on the wheel in Figure P10.31 about the axle through O if $a = 10.0$ cm and $b = 25.0$ cm.

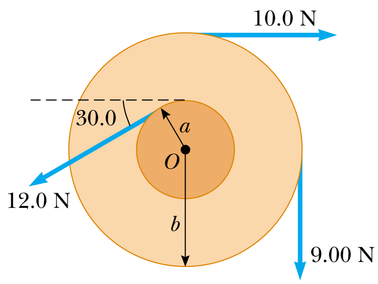


Figure P10.31

32. The tires of a 1 500-kg car are 0.600 m in diameter, and the coefficients of friction with the road surface are $\mu_s = 0.800$ and $\mu_k = 0.600$. Assuming that the weight is evenly distributed on the four wheels, calculate the maximum torque that can be exerted by the engine on a driving wheel without spinning the wheel. If you wish, you may assume the car is at rest.

33. Suppose the car in Problem 32 has a disk brake system. Each wheel is slowed by the friction force between a single brake pad and the disk-shaped rotor. On this particular car, the brake pad contacts the rotor at an average distance of 22.0 cm from the axis. The coefficients of friction between the brake pad and the disk are $\mu_s = 0.600$ and $\mu_k = 0.500$. Calculate the normal force that the pad must apply to the rotor in order to slow the car as quickly as possible.

Section 10.7 Relationship between Torque and Angular Acceleration

34. A grinding wheel is in the form of a uniform solid disk of radius 7.00 cm and mass 2.00 kg. It starts from rest and accelerates uniformly under the action of the constant torque of 0.600 N · m that the motor exerts on the wheel. (a) How long does the wheel take to reach its final operating speed of 1 200 rev/min? (b) Through how many revolutions does it turn while accelerating?

35. A model airplane with mass 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the linear acceleration of the airplane tangent to its flight path.

36. The combination of an applied force and a friction force produces a constant total torque of 36.0 N · m on a wheel rotating about a fixed axis. The applied force acts for 6.00 s. During this time the angular speed of the wheel increases from 0 to 10.0 rad/s. The applied force is then removed, and the wheel comes to rest in 60.0 s. Find (a) the moment of inertia of the wheel, (b) the magnitude of the frictional torque, and (c) the total number of revolutions of the wheel.

37. A block of mass $m_1 = 2.00$ kg and a block of mass $m_2 = 6.00$ kg are connected by a massless string over a pulley in the shape of a solid disk having radius $R = 0.250$ m and mass $M = 10.0$ kg. These blocks are allowed to move on a fixed block-wedge of angle $\theta = 30.0°$ as in Figure P10.37. The coefficient of kinetic friction is 0.360 for both blocks. Draw free-body diagrams of both blocks and of the pulley. Determine (a) the acceleration of the two blocks and (b) the tensions in the string on both sides of the pulley.

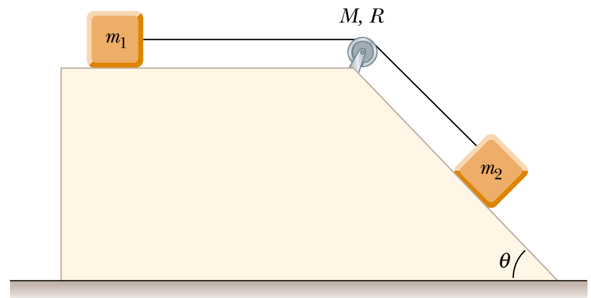


Figure P10.37

38. A potter's wheel—a thick stone disk of radius 0.500 m and mass 100 kg—is freely rotating at 50.0 rev/min. The potter can stop the wheel in 6.00 s by pressing a wet rag against the rim and exerting a radially inward force of 70.0 N. Find the effective coefficient of kinetic friction between wheel and rag.

39. An electric motor turns a flywheel through a drive belt that joins a pulley on the motor and a pulley that is rigidly attached to the flywheel, as shown in Figure P10.39. The flywheel is a solid disk with a mass of 80.0 kg and a diameter of 1.25 m. It turns on a frictionless axle. Its pulley has much smaller mass and a radius of 0.230 m. If the tension in the upper (taut) segment of the belt is 135 N and the flywheel has a clockwise angular acceleration of 1.67 rad/s^2, find the tension in the lower (slack) segment of the belt.

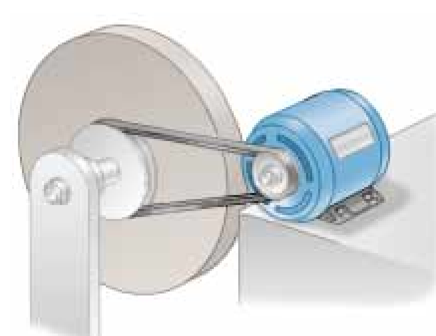


Figure P10.39

Section 10.8 Work, Power, and Energy in Rotational Motion

40. Big Ben, the Parliament tower clock in London, has an hour hand 2.70 m long with a mass of 60.0 kg, and a minute hand 4.50 m long with a mass of 100 kg (Fig. P10.40). Calculate the total rotational kinetic energy of the two hands about the axis of rotation. (You may model the hands as long, thin rods.)

John Lawrence /Getty

Figure P10.40 Problems 40 and 74.

41. In a city with an air-pollution problem, a bus has no combustion engine. It runs on energy drawn from a large, rapidly rotating flywheel under the floor of the bus. The flywheel is spun up to its maximum rotation rate of 4 000 rev/min by an electric motor at the bus terminal. Every time the bus speeds up, the flywheel slows down slightly. The bus is equipped with regenerative braking so that the flywheel can speed up when the bus slows down. The flywheel is a uniform solid cylinder with mass 1 600 kg and radius 0.650 m. The bus body does work against air resistance and rolling resistance at the average rate of 18.0 hp as it travels with an average speed of 40.0 km/h. How far can the bus travel before the flywheel has to be spun up to speed again?

42. The top in Figure P10.42 has a moment of inertia of $4.00 \times 10^{-4}\ \text{kg} \cdot \text{m}^2$ and is initially at rest. It is free to rotate about the stationary axis AA'. A string, wrapped around a peg along the axis of the top, is pulled in such a manner as to maintain a constant tension of 5.57 N. If the string does not slip while it is unwound from the peg, what is the angular speed of the top after 80.0 cm of string has been pulled off the peg?

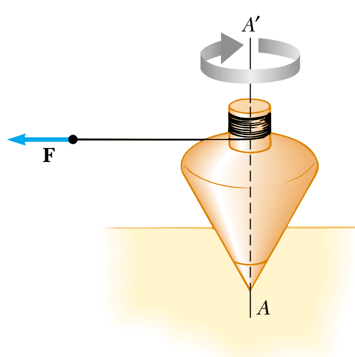


Figure P10.42

43. In Figure P10.43 the sliding block has a mass of 0.850 kg, the counterweight has a mass of 0.420 kg, and the pulley is a hollow cylinder with a mass of 0.350 kg, an inner radius of 0.020 0 m, and an outer radius of 0.030 0 m. The coefficient of kinetic friction between the block and the horizontal surface is 0.250. The pulley turns without friction on its axle. The light cord does not stretch and does not slip on the pulley. The block has a velocity of 0.820 m/s toward the pulley when it passes through a photogate. (a) Use energy methods to predict its speed after it has moved to a second photogate, 0.700 m away. (b) Find the angular speed of the pulley at the same moment.

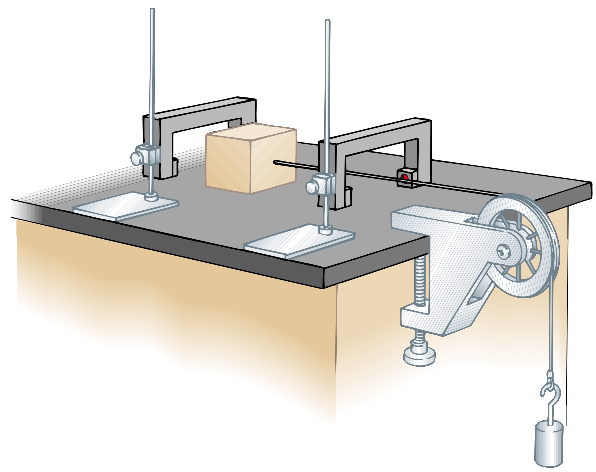


Figure P10.43

44. A cylindrical rod 24.0 cm long with mass 1.20 kg and radius 1.50 cm has a ball of diameter 8.00 cm and mass 2.00 kg attached to one end. The arrangement is originally vertical and stationary, with the ball at the top. The system is free to pivot about the bottom end of the rod after being given a slight nudge. (a) After the rod rotates through ninety degrees, what is its rotational kinetic energy? (b) What is the angular speed of the rod and ball? (c) What is the linear speed of the ball? (d) How does this compare to the speed if the ball had fallen freely through the same distance of 28 cm?

$\boxed{\textbf{45.}}$ An object with a weight of 50.0 N is attached to the free end of a light string wrapped around a reel of radius 0.250 m and mass 3.00 kg. The reel is a solid disk, free to rotate in a vertical plane about the horizontal axis passing through its center. The suspended object is released 6.00 m above the floor. (a) Determine the tension in the string, the acceleration of the object, and the speed with which the object hits the floor. (b) Verify your last answer by using the principle of conservation of energy to find the speed with which the object hits the floor.

46. A 15.0-kg object and a 10.0-kg object are suspended, joined by a cord that passes over a pulley with a radius of 10.0 cm and a mass of 3.00 kg (Fig. P10.46). The cord has a negligible mass and does not slip on the pulley. The pulley rotates on its axis without friction. The objects start from rest 3.00 m apart. Treat the pulley as a uniform disk, and determine the speeds of the two objects as they pass each other.

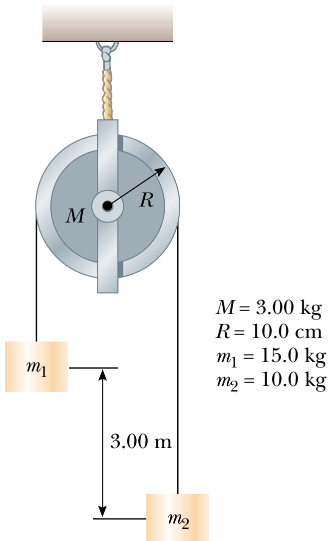


Figure P10.46

47. This problem describes one experimental method for determining the moment of inertia of an irregularly shaped object such as the payload for a satellite. Figure P10.47 shows a counterweight of mass m suspended by a cord wound around a spool of radius r, forming part of a turntable supporting the object. The turntable can rotate without friction. When the counterweight is released from rest, it descends through a distance h, acquiring a speed v. Show that the moment of inertia I of the rotating apparatus (including the turntable) is $mr^2(2gh/v^2 - 1)$.

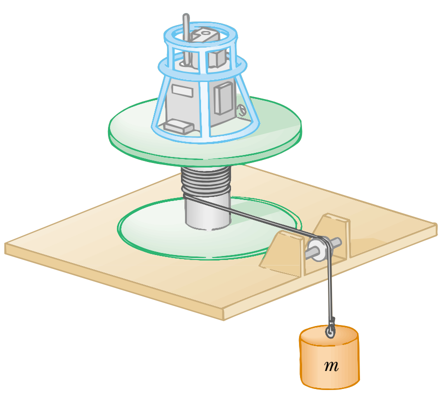


Figure P10.47

48. A horizontal 800-N merry-go-round is a solid disk of radius 1.50 m, started from rest by a constant horizontal force of 50.0 N applied tangentially to the edge of the disk. Find the kinetic energy of the disk after 3.00 s.

49. (a) A uniform solid disk of radius R and mass M is free to rotate on a frictionless pivot through a point on its rim (Fig. P10.49). If the disk is released from rest in the position shown by the blue circle, what is the speed of its center of mass when the disk reaches the position indicated by the dashed circle? (b) What is the speed of the lowest point on the disk in the dashed position? (c) **What If?** Repeat part (a) using a uniform hoop.

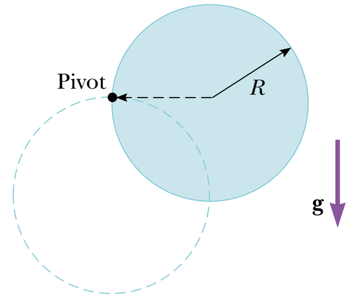


Figure P10.49

50. The head of a grass string trimmer has 100 g of cord wound in a light cylindrical spool with inside diameter 3.00 cm and outside diameter 18.0 cm, as in Figure P10.50. The cord has a linear density of 10.0 g/m. A single strand of the cord extends 16.0 cm from the outer edge of the spool. (a) When switched on, the trimmer speeds up from 0 to 2 500 rev/min in 0.215 s. (a) What average power is delivered to the head by the trimmer motor while it is accelerating? (b) When the trimmer is cutting grass, it spins at 2 000 rev/min and the grass exerts an average tangential force of 7.65 N on the outer end of the cord, which is still at a radial distance of 16.0 cm from the outer edge of the spool. What is the power delivered to the head under load?

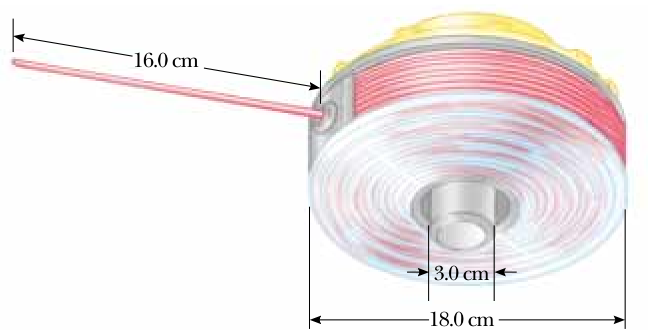


Figure P10.50

Section 10.9 Rolling Motion of a Rigid Object

51. A cylinder of mass 10.0 kg rolls without slipping on a horizontal surface. At the instant its center of mass has a speed of 10.0 m/s, determine (a) the translational kinetic energy of its center of mass, (b) the rotational kinetic energy about its center of mass, and (c) its total energy.

52. A bowling ball has mass M, radius R, and a moment of inertia of $\frac{2}{5}MR^2$. If it starts from rest, how much work must be done on it to set it rolling without slipping at a linear speed v? Express the work in terms of M and v.

53. (a) Determine the acceleration of the center of mass of a uniform solid disk rolling down an incline making angle θ with the horizontal. Compare this acceleration with that of a uniform hoop. (b) What is the minimum coefficient of friction required to maintain pure rolling motion for the disk?

54. A uniform solid disk and a uniform hoop are placed side by side at the top of an incline of height h. If they are released from rest and roll without slipping, which object reaches the bottom first? Verify your answer by calculating their speeds when they reach the bottom in terms of h.

55. A metal can containing condensed mushroom soup has mass 215 g, height 10.8 cm, and diameter 6.38 cm. It is placed at rest on its side at the top of a 3.00-m-long incline that is at 25.0° to the horizontal, and it is then released to roll straight down. Assuming mechanical energy conservation, calculate the moment of inertia of the can if it takes 1.50 s to reach the bottom of the incline. Which pieces of data, if any, are unnecessary for calculating the solution?

56. A tennis ball is a hollow sphere with a thin wall. It is set rolling without slipping at 4.03 m/s on a horizontal section of a track, as shown in Figure P10.56. It rolls around the inside of a vertical circular loop 90.0 cm in diameter and finally leaves the track at a point 20.0 cm below the horizontal section. (a) Find the speed of the ball at the top of the loop. Demonstrate that it will not fall from the track. (b) Find its speed as it leaves the track. **What If?** (c) Suppose that static friction between ball and track were negligible, so that the ball slid instead of rolling. Would its speed then be higher, lower, or the same at the top of the loop? Explain.

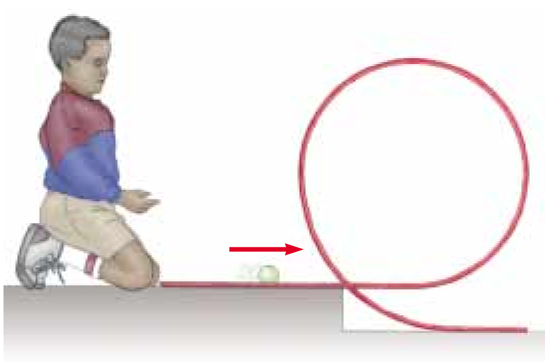
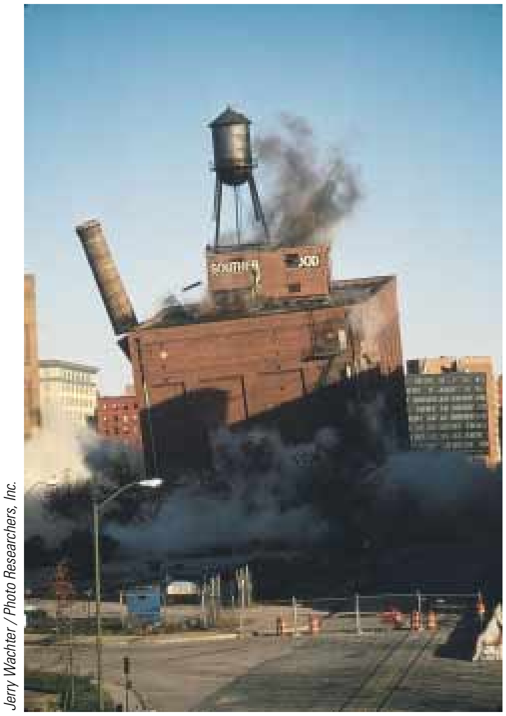


Figure P10.56

Additional Problems

57. As in Figure P10.57, toppling chimneys often break apart in mid-fall because the mortar between the bricks cannot withstand much shear stress. As the chimney begins to fall, shear forces must act on the topmost sections to accelerate them tangentially so that they can keep up with the rotation of the lower part of the stack. For simplicity, let us model the chimney as a uniform rod of length ℓ pivoted at the lower end. The rod starts at rest in a vertical position (with the frictionless pivot at the bottom) and falls over under the influence of gravity. What fraction of the length of the rod has a tangential acceleration greater than $g \sin \theta$, where θ is the angle the chimney makes with the vertical axis?



Jerry Wachter / Photo Researchers, Inc.

Figure P10.57 A building demolition site in Baltimore, MD. At the left is a chimney, mostly concealed by the building, that has broken apart on its way down. Compare with Figure 10.19.

58. **Review problem.** A mixing beater consists of three thin rods, each 10.0 cm long. The rods diverge from a central hub, separated from each other by 120°, and all turn in the same plane. A ball is attached to the end of each rod. Each ball has cross-sectional area 4.00 cm^2 and is so shaped that it has a drag coefficient of 0.600. Calculate the power input required to spin the beater at 1 000 rev/min (a) in air and (b) in water.

$\boxed{59.}$ A 4.00-m length of light nylon cord is wound around a uniform cylindrical spool of radius 0.500 m and mass 1.00 kg. The spool is mounted on a frictionless axle and is initially at rest. The cord is pulled from the spool with a constant acceleration of magnitude 2.50 m/s^2. (a) How much work has been done on the spool when it reaches an angular speed of 8.00 rad/s? (b) Assuming there is enough cord on the spool, how long does it take the spool to reach this angular speed? (c) Is there enough cord on the spool?

60. A videotape cassette contains two spools, each of radius r_s, on which the tape is wound. As the tape unwinds from the first spool, it winds around the second spool. The tape moves at constant linear speed v past the heads between the spools. When all the tape is on the first spool, the tape has an outer radius r_t. Let r represent the outer radius of the tape on the first spool at any instant while the tape is being played. (a) Show that at any instant the angular speeds of the two spools are

$$\omega_1 = v/r \qquad \text{and} \qquad \omega_2 = v/(r_s^{\,2} + r_t^{\,2} - r^2)^{1/2}$$

(b) Show that these expressions predict the correct maximum and minimum values for the angular speeds of the two spools.

61. A long uniform rod of length L and mass M is pivoted about a horizontal, frictionless pin through one end. The rod is released from rest in a vertical position, as shown in Figure P10.61. At the instant the rod is horizontal, find (a) its angular speed, (b) the magnitude of its angular acceleration, (c) the x and y components of the acceleration of its center of mass, and (d) the components of the reaction force at the pivot.

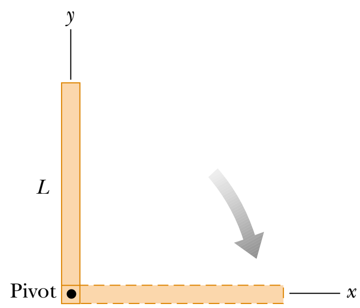


Figure P10.61

62. A shaft is turning at 65.0 rad/s at time $t = 0$. Thereafter, its angular acceleration is given by

$$\alpha = -10.0 \text{ rad/s}^2 - 5.00t \text{ rad/s}^3,$$

where t is the elapsed time. (a) Find its angular speed at $t = 3.00$ s. (b) How far does it turn in these 3 s?

63. A bicycle is turned upside down while its owner repairs a flat tire. A friend spins the other wheel, of radius 0.381 m, and observes that drops of water fly off tangentially. She measures the height reached by drops moving vertically (Fig. P10.63). A drop that breaks loose from the tire on one turn rises $h = 54.0$ cm above the tangent point. A drop that breaks loose on the next turn rises 51.0 cm above the tangent point. The height to which the drops rise decreases because the angular speed of the wheel decreases. From this information, determine the magnitude of the average angular acceleration of the wheel.

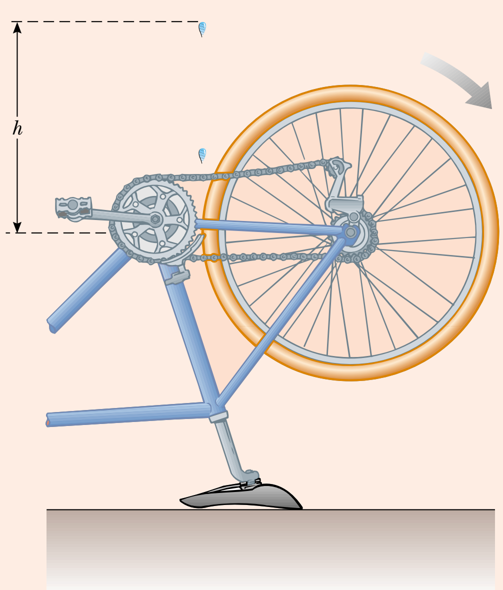


Figure P10.63 Problems 63 and 64.

64. A bicycle is turned upside down while its owner repairs a flat tire. A friend spins the other wheel, of radius R, and observes that drops of water fly off tangentially. She measures the height reached by drops moving vertically (Fig. P10.63). A drop that breaks loose from the tire on one turn rises a distance h_1 above the tangent point. A drop that breaks loose on the next turn rises a distance $h_2 < h_1$ above the tangent point. The height to which the drops rise decreases because the angular speed of the wheel decreases. From this information, determine the magnitude of the average angular acceleration of the wheel.

65. A cord is wrapped around a pulley of mass m and radius r. The free end of the cord is connected to a block of mass M. The block starts from rest and then slides down an incline that makes an angle θ with the horizontal. The coefficient of kinetic friction between block and incline is μ. (a) Use energy methods to show that the block's speed as a function of position d down the incline is

$$v = \sqrt{\frac{4gdM(\sin\theta - \mu\cos\theta)}{m + 2M}}$$

(b) Find the magnitude of the acceleration of the block in terms of μ, m, M, g, and θ.

66. (a) What is the rotational kinetic energy of the Earth about its spin axis? Model the Earth as a uniform sphere and use data from the endpapers. (b) The rotational kinetic energy of the Earth is decreasing steadily because of tidal friction. Find the change in one day, assuming that the rotational period decreases by 10.0 μs each year.

67. Due to a gravitational torque exerted by the Moon on the Earth, our planet's rotation period slows at a rate on the order of 1 ms/century. (a) Determine the order of magnitude of the Earth's angular acceleration. (b) Find the order of magnitude of the torque. (c) Find the order of magnitude of the size of the wrench an ordinary person would need to exert such a torque, as in Figure P10.67. Assume the person can brace his feet against a solid firmament.

Figure P10.67

68. The speed of a moving bullet can be determined by allowing the bullet to pass through two rotating paper disks mounted a distance d apart on the same axle (Fig. P10.68). From the angular displacement $\Delta\theta$ of the two bullet holes in the disks and the rotational speed of the disks, we can determine the speed v of the bullet. Find the bullet speed for the following data: $d = 80$ cm, $\omega = 900$ rev/min, and $\Delta\theta = 31.0°$.

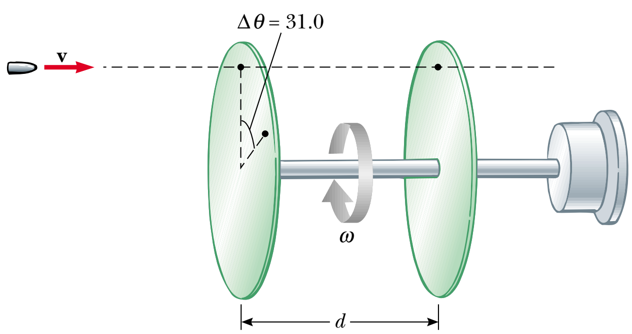


Figure P10.68

69. A uniform, hollow, cylindrical spool has inside radius $R/2$, outside radius R, and mass M (Fig. P10.69). It is mounted so that it rotates on a fixed horizontal axle. A counterweight of mass m is connected to the end of a string wound around the spool. The counterweight falls from rest at $t = 0$ to a position y at time t. Show that the torque due to the friction forces between spool and axle is

$$\tau_f = R\left[m\left(g - \frac{2y}{t^2}\right) - M\frac{5y}{4t^2}\right]$$

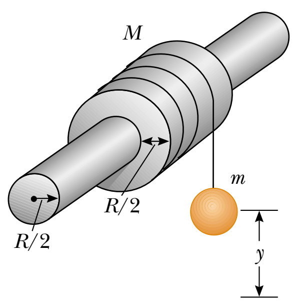


Figure P10.69

70. The reel shown in Figure P10.70 has radius R and moment of inertia I. One end of the block of mass m is connected to a spring of force constant k, and the other end is fastened to a cord wrapped around the reel. The reel axle and the incline are frictionless. The reel is wound counterclockwise so that the spring stretches a distance d from its unstretched position and is then released from rest. (a) Find the angular speed of the reel when the spring is again unstretched. (b) Evaluate the angular speed numerically at this point if $I = 1.00$ kg · m^2, $R = 0.300$ m, $k = 50.0$ N/m, $m = 0.500$ kg, $d = 0.200$ m, and $\theta = 37.0°$.

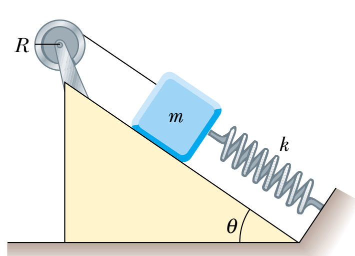


Figure P10.70

71. Two blocks, as shown in Figure P10.71, are connected by a string of negligible mass passing over a pulley of radius 0.250 m and moment of inertia I. The block on the frictionless incline is moving up with a constant acceleration of 2.00 m/s^2. (a) Determine T_1 and T_2, the tensions in the two parts of the string. (b) Find the moment of inertia of the pulley.

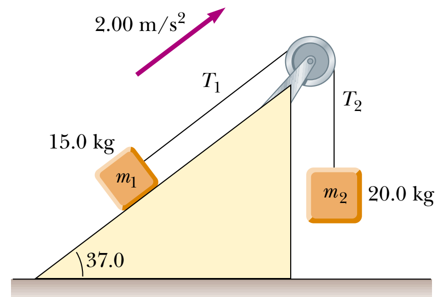


Figure P10.71

72. A common demonstration, illustrated in Figure P10.72, consists of a ball resting at one end of a uniform board of length ℓ, hinged at the other end, and elevated at an angle θ. A light cup is attached to the board at r_c so that it will catch the ball when the support stick is suddenly removed. (a) Show that the ball will lag behind the falling board when θ is less than 35.3°. (b) If the board is 1.00 m long and is supported at this limiting angle, show that the cup must be 18.4 cm from the moving end.

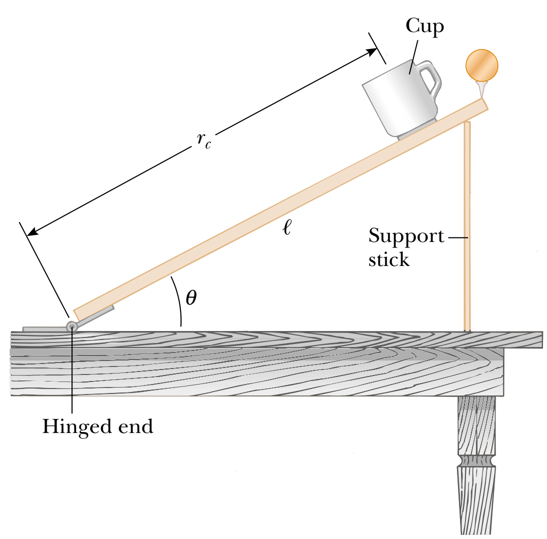


Figure P10.72

73. As a result of friction, the angular speed of a wheel changes with time according to

$$\frac{d\theta}{dt} = \omega_0 e^{-\sigma t}$$

where ω_0 and σ are constants. The angular speed changes from 3.50 rad/s at $t = 0$ to 2.00 rad/s at $t = 9.30$ s. Use this information to determine σ and ω_0. Then determine (a) the magnitude of the angular acceleration at $t = 3.00$ s, (b) the number of revolutions the wheel makes in the first 2.50 s, and (c) the number of revolutions it makes before coming to rest.

74. The hour hand and the minute hand of Big Ben, the Parliament tower clock in London, are 2.70 m and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively (see Figure P10.40). (a) Determine the total torque due to the weight of these hands about the axis of rotation when the time reads (i) 3:00 (ii) 5:15 (iii) 6:00 (iv) 8:20 (v) 9:45. (You may model the hands as long, thin uniform rods.) (b) Determine all times when the total torque about the axis of rotation is zero. Determine the times to the nearest second, solving a transcendental equation numerically.

75. (a) Without the wheels, a bicycle frame has a mass of 8.44 kg. Each of the wheels can be roughly modeled as a uniform solid disk with a mass of 0.820 kg and a radius of 0.343 m. Find the kinetic energy of the whole bicycle when it is moving forward at 3.35 m/s. (b) Before the invention of a wheel turning on an axle, ancient people moved heavy loads by placing rollers under them. (Modern people use rollers too. Any hardware store will sell you a roller bearing for a lazy susan.) A stone block of mass 844 kg moves forward at 0.335 m/s, supported by two uniform cylindrical tree trunks, each of mass 82.0 kg and radius 0.343 m. No slipping occurs between the block and the rollers or between the rollers and the ground. Find the total kinetic energy of the moving objects.

76. A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl with much larger radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping (Fig. P10.76). Determine the angular speed of the sphere when it reaches the bottom of the bowl.

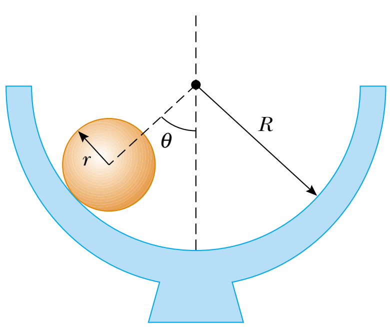


Figure P10.76

77. A string is wound around a uniform disk of radius R and mass M. The disk is released from rest with the string vertical and its top end tied to a fixed bar (Fig. P10.77). Show that (a) the tension in the string is one third of the weight of the disk, (b) the magnitude of the acceleration of the center of mass is $2g/3$, and (c) the speed of the center of mass is $(4gh/3)^{1/2}$ after the disk has descended through distance h. Verify your answer to (c) using the energy approach.

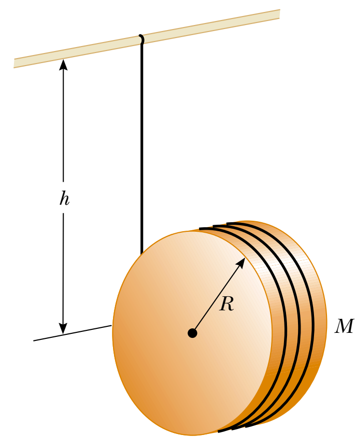


Figure P10.77

78. A constant horizontal force $\mathbf{F}$ is applied to a lawn roller in the form of a uniform solid cylinder of radius R and mass M (Fig. P10.78). If the roller rolls without slipping on the horizontal surface, show that (a) the acceleration of the center of mass is $2\mathbf{F}/3M$ and (b) the minimum coefficient of friction necessary to prevent slipping is $F/3Mg$. (*Hint:* Take the torque with respect to the center of mass.)

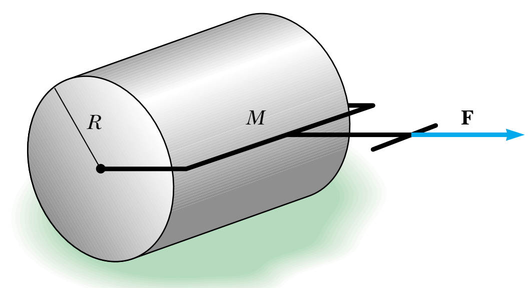


Figure P10.78

79. A solid sphere of mass m and radius r rolls without slipping along the track shown in Figure P10.79. It starts from rest with the lowest point of the sphere at height h above the bottom of the loop of radius R, much larger than r. (a) What is the minimum value of h (in terms of R) such that the sphere completes the loop? (b) What are the force components on the sphere at the point P if $h = 3R$?

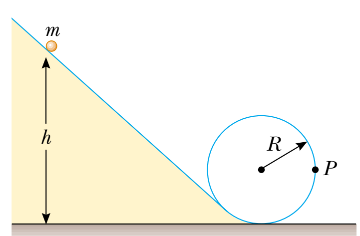


Figure P10.79

80. A thin rod of mass 0.630 kg and length 1.24 m is at rest, hanging vertically from a strong fixed hinge at its top end. Suddenly a horizontal impulsive force $(14.7\hat{\mathbf{i}})$ N is applied to it. (a) Suppose the force acts at the bottom end of the rod. Find the acceleration of its center of mass and the horizontal force the hinge exerts. (b) Suppose the force acts at the midpoint of the rod. Find the acceleration of this point and the horizontal hinge reaction. (c) Where can the impulse be applied so that the hinge will exert no horizontal force? This point is called the *center of percussion.*

81. A bowler releases a bowling ball with no spin, sending it sliding straight down the alley toward the pins. The ball continues to slide for a distance of what order of magnitude, before its motion becomes rolling without slipping? State the quantities you take as data, the values you measure or estimate for them, and your reasoning.

82. Following Thanksgiving dinner your uncle falls into a deep sleep, sitting straight up facing the television set. A naughty grandchild balances a small spherical grape at the top of his bald head, which itself has the shape of a sphere. After all the children have had time to giggle, the grape starts from rest and rolls down without slipping. It will leave contact with your uncle's scalp when the radial line joining it to the center of curvature makes what angle with the vertical?

83. (a) A thin rod of length h and mass M is held vertically with its lower end resting on a frictionless horizontal surface. The rod is then released to fall freely. Determine the speed of its center of mass just before it hits the horizontal surface. (b) **What If?** Now suppose the rod has a fixed pivot at its lower end. Determine the speed of the rod's center of mass just before it hits the surface.

84. A large, cylindrical roll of tissue paper of initial radius R lies on a long, horizontal surface with the outside end of the paper nailed to the surface. The roll is given a slight shove ($v_i \approx 0$) and commences to unroll. Assume the roll has a uniform density and that mechanical energy is conserved in the process. (a) Determine the speed of the center of mass of the roll when its radius has diminished to r. (b) Calculate a numerical value for this speed at $r = 1.00$ mm, assuming $R = 6.00$ m. (c) **What If?** What happens to the energy of the system when the paper is completely unrolled?

85. A spool of wire of mass M and radius R is unwound under a constant force $\mathbf{F}$ (Fig. P10.85). Assuming the spool is a uniform solid cylinder that doesn't slip, show that (a) the acceleration of the center of mass is $4\mathbf{F}/3M$ and (b) the force of friction is to the *right* and equal in magnitude to $F/3$. (c) If the cylinder starts from rest and rolls without slipping, what is the speed of its center of mass after it has rolled through a distance d?

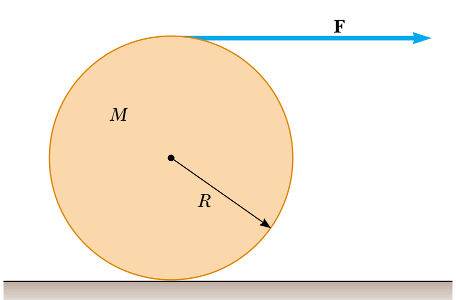


Figure P10.85

86. A plank with a mass $M = 6.00$ kg rides on top of two identical solid cylindrical rollers that have $R = 5.00$ cm and $m = 2.00$ kg (Fig. P10.86). The plank is pulled by a constant horizontal force $\mathbf{F}$ of magnitude 6.00 N applied to the end of the plank and perpendicular to the axes of the cylinders (which are parallel). The cylinders roll without slipping on a flat surface. There is also no slipping between the cylinders and the plank. (a) Find the acceleration of the plank and of the rollers. (b) What friction forces are acting?

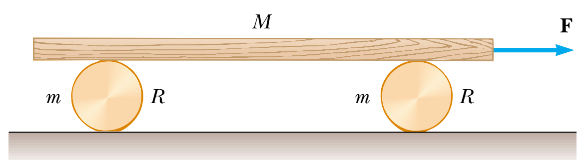


Figure P10.86

87. A spool of wire rests on a horizontal surface as in Figure P10.87. As the wire is pulled, the spool does not slip at the contact point P. On separate trials, each one of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ is applied to the spool. For each one of these forces, determine the direction the spool will roll. Note that the line of action of $\mathbf{F}_2$ passes through P.

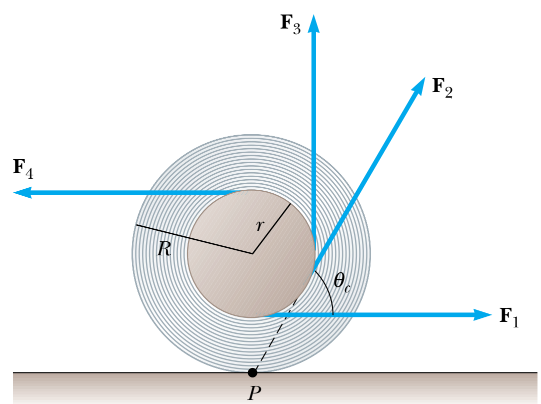


Figure P10.87 Problems 87 and 88.

88. Refer to Problem 87 and Figure P10.87. The spool of wire has an inner radius r and an outer radius R. The angle θ between the applied force and the horizontal can be varied. Show that the critical angle for which the spool does not roll is given by

$$\cos\theta_c = \frac{r}{R}$$

If the wire is held at this angle and the force increased, the spool will remain stationary until it slips along the floor.

89. In a demonstration known as the ballistics cart, a ball is projected vertically upward from a cart moving with constant velocity along the horizontal direction. The ball lands in the catching cup of the cart because both the cart a ball have the same horizontal component of velocity. **What If?** Now consider a ballistics cart on an incline making an angle θ with the horizontal as in Figure P10.89. The cart (including wheels) has a mass M and the moment of inertia of each of the two wheels is $mR^2/2$. (a) Using conservation of energy (assuming no friction between cart and axles) and assuming pure rolling motion (no slipping), show that the acceleration of the cart along the incline is

$$a_x = \left(\frac{M}{M + 2m}\right) g \sin\theta$$

(b) Note that the x component of acceleration of the ball released by the cart is $g\sin\theta$. Thus, the x component of the cart's acceleration is *smaller* than that of the ball by the factor $M/(M + 2m)$. Use this fact and kinematic equations to show that the ball overshoots the cart by an amount Δx, where

$$\Delta x = \left(\frac{4m}{M + 2m}\right)\left(\frac{\sin\theta}{\cos^2\theta}\right)\frac{v_{yi}^2}{g}$$

and v_{yi} is the initial speed of the ball imparted to it by the spring in the cart. (c) Show that the distance d that the ball travels measured along the incline is

$$d = \frac{2v_{yi}^2}{g}\,\frac{\sin\theta}{\cos^2\theta}$$

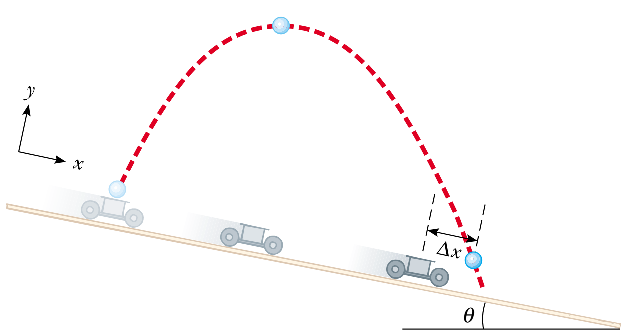


Figure P10.89

90. A spool of thread consists of a cylinder of radius R_1 with end caps of radius R_2 as in the end view shown in Figure P10.90. The mass of the spool, including the thread, is m and its moment of inertia about an axis through its center is I. The spool is placed on a rough horizontal surface so that it rolls without slipping when a force $\mathbf{T}$ acting to the right is applied to the free end of the thread. Show that the magnitude of the friction force exerted by the surface on the spool is given by

$$f = \left(\frac{I + mR_1R_2}{I + mR_2^2}\right) T$$

Determine the direction of the force of friction.

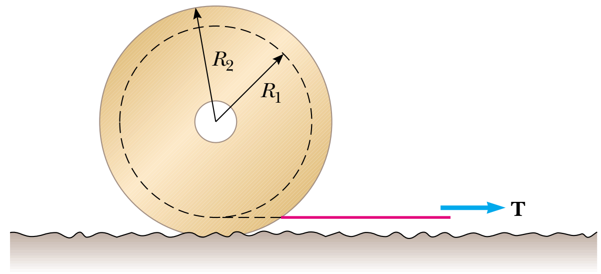


Figure P10.90

Answers to Quick Quizzes

10.1 (c). For a rotation of more than 180°, the angular displacement must be larger than $\pi = 3.14$ rad. The angular displacements in the three choices are (a) 6 rad − 3 rad = 3 rad (b) 1 rad − (− 1) rad = 2 rad (c) 5 rad − 1 rad = 4 rad.

10.2 (b). Because all angular displacements occur in the same time interval, the displacement with the lowest value will be associated with the lowest average angular speed.

10.3 (b). The fact that ω is negative indicates that we are dealing with an object that is rotating in the clockwise direction. We also know that when $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are antiparallel, ω must be decreasing—the object is slowing down. Therefore, the object is spinning more and more slowly (with less and less angular speed) in the clockwise, or negative, direction.

10.4 (b). In Equation 10.8, both the initial and final angular speeds are the same in all three cases. As a result, the angular acceleration is inversely proportional to the angular displacement. Thus, the highest angular acceleration is associated with the lowest angular displacement.

10.5 (b). The system of the platform, Andy, and Charlie is a rigid object, so all points on the rigid object have the same angular speed.

10.6 (a). The tangential speed is proportional to the radial distance from the rotation axis.

10.7 (a). Almost all of the mass of the pipe is at the same distance from the rotation axis, so it has a larger moment of inertia than the solid cylinder.

10.8 (b). The fatter handle of the screwdriver gives you a larger moment arm and increases the torque that you can apply with a given force from your hand.

10.9 (a). The longer handle of the wrench gives you a larger moment arm and increases the torque that you can apply with a given force from your hand.

10.10 (b). With twice the moment of inertia and the same frictional torque, there is half the angular acceleration. With half the angular acceleration, it will require twice as long to change the speed to zero.

10.11 (d). When the rod is attached at its end, it offers four times as much moment of inertia as when attached in the center (see Table 10.2). Because the rotational kinetic energy of the rod depends on the square of the angular speed, the same work will result in half of the angular speed.

10.12 (b). All of the gravitational potential energy of the box–Earth system is transformed to kinetic energy of translation. For the ball, some of the gravitational potential energy of the ball–Earth system is transformed to rotational kinetic energy, leaving less for translational kinetic energy, so the ball moves downhill more slowly than the box does.

10.13 (c). In Equation 10.30, I_{CM} for a sphere is $\frac{2}{5}MR^2$. Thus, MR^2 will cancel and the remaining expression on the right-hand side of the equation is independent of mass and radius.

10.14 (a). The moment of inertia of the hollow sphere B is larger than that of sphere A. As a result, Equation 10.30 tells us that the center of mass of sphere B will have a smaller speed, so sphere A should arrive first.